2.1 Areas between Curves

Math 1700

University of Manitoba

2024

Outline

1. Area of a Region between Two Curves
2. Areas of Compound Regions
3. Regions Defined with Respect to y

Learning Objectives

Objective 1

Determine the area of a region between two curves by integrating with respect to x.

Objective 2

Find the area of a compound region.

Objective 3

Determine the area of a region between two curves by integrating with respect to y.

Introduction

Introduction to Integration

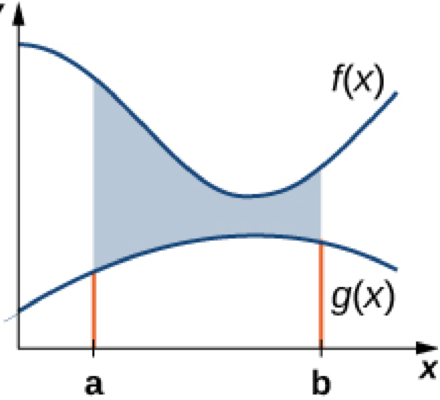
In Introduction to Integration, we developed the concept of the definite integral to calculate the area below a curve on a given interval.

Expanding the Idea

In this section, we expand that idea to calculate the area of more complex regions.

Area of a Region between Two Curves

Let $f(x)$ and $g(x)$ be continuous functions over an interval $[a, b]$ such that $f(x) \geq g(x)$ on $[a, b]$. We want to find the area between the graphs of the functions.



Finding the Area between Two Curves

Theorem

Let $f(x)$ and $g(x)$ be continuous functions such that $f(x) \geq g(x)$ over an interval $[a, b]$. Let R denote the region bounded above by the graph of $f(x)$, below by the graph of $g(x)$, and on the left and right by the lines $x = a$ and $x = b$, respectively. Then, the area of R is given by:

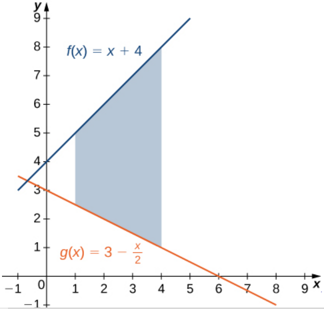
$$A = \int_a^b [f(x) - g(x)]\, dx$$

Finding the Area of a Region between Two Curves 1

Example: If R is the region bounded above by the graph of the function $f(x) = x + 4$ and below by the graph of the function $g(x) = 3 - \frac{x}{2}$ over the interval $[1, 4]$, find the area of region R.

Solution

$$A = \int_1^4 [(x+4) - (3 - \frac{x}{2})]\, dx$$
$$= \int_1^4 [\frac{3x}{2} + 1]\, dx$$
$$= [\frac{3x^2}{4} + x]\Big|_1^4$$
$$= \frac{57}{4} \text{ units}^2$$

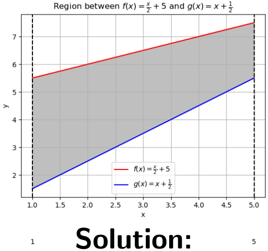


Area of Region Bounded by Two Curves

Example: If R is the region bounded by the graphs of the functions $f(x) = \frac{x}{2} + 5$ and $g(x) = x + \frac{1}{2}$ over the interval $[1, 5]$, find the area of region R.

Area of Region Bounded by Two Curves

Example: If R is the region bounded by the graphs of the functions $f(x) = \frac{x}{2} + 5$ and $g(x) = x + \frac{1}{2}$ over the interval $[1, 5]$, find the area of region R.



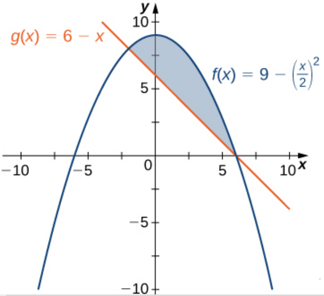
Solution:

$$A = \int_1^5 \left(\left(\frac{x}{2} + 5 \right) - \left(x + \frac{1}{2} \right) \right) \, dx = \int_1^5 \left(\frac{x}{2} + 5 - x - \frac{1}{2} \right) \, dx$$

$$= \int_1^5 \left(-\frac{1}{2}x + \frac{9}{2} \right) \, dx = \left[-\frac{1}{4}x^2 + \frac{9}{2}x \right]_1^5 = 12 \text{ units}^2$$

Finding the Area of a Region between Two Curves 2

Example: If R is the region bounded above by the graph of the function $f(x) = 9 - \left(\frac{x}{2}\right)^2$ and below by the graph of the function $g(x) = 6 - x$, find the area of region R.



Finding the Area of a Region between Two Curves 2

Example: If R is the region bounded above by the graph of the function $f(x) = 9 - \left(\frac{x}{2}\right)^2$ and below by the graph of the function $g(x) = 6 - x$, find the area of region R.

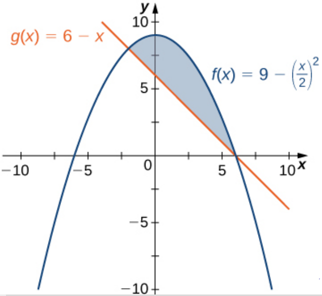
Solution (Find the domain?)

$$A = \int_{-2}^{6} [9 - \left(\frac{x}{2}\right)^2 - (6 - x)]\, dx$$

$$= \int_{-2}^{6} [3 - \frac{x^2}{4} + x]\, dx$$

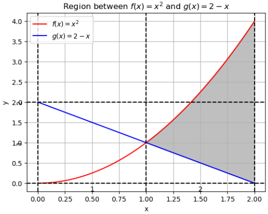
$$= [3x - \frac{x^3}{12} + \frac{x^2}{2}]\Big|_{-2}^{6}$$

$$= \frac{64}{3} \text{ units}^2$$



Areas of Compound Regions

Example: Consider the region bounded by the graphs of the functions $f(x) = x^2$ and $g(x) = 2 - x$ over the interval $[0, 2]$. Find the area of region R.



Solution

$$A = \int_0^1 (x^2 - (2 - x))\,dx + \int_1^2 ((2 - x) - x^2)\,dx$$
$$= \frac{1}{3} + \frac{1}{2} = \frac{5}{6} \text{ units}^2$$

Finding the Area of a Region between Intersecting Curves

So far, we have required $f(x) \geq g(x)$

Finding the Area of a Region between Intersecting Curves

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Let $f(x)$ and $g(x)$ be continuous functions over an interval $[a, b]$. Let R denote the region between the graphs of $f(x)$ and $g(x)$, and be bounded on the left and right by the lines $x = a$ and $x = b$, respectively. Then, the area of R is given by

$$A = \int_a^b |f(x) - g(x)|\, dx.$$

Example

If R is the region between the graphs of the functions $f(x) = \sin(x)$ and $g(x) = \cos(x)$ over the interval $[0, \pi]$, find the area of region R.

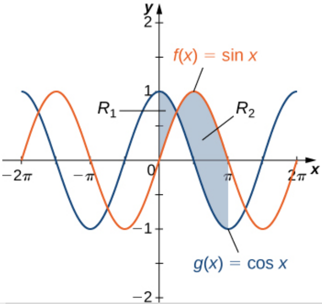
Solution Part 1: $A = \int_0^\pi |f(x) - g(x)|\, dx$

The graphs of the functions intersect at $x = \frac{\pi}{4}$. For $x \in \left[0, \frac{\pi}{4}\right]$, $\cos(x) \geq \sin(x)$, so

$|f(x) - g(x)| = |\sin(x) - \cos(x)| = \cos(x) - \sin(x)$.

On the other hand, for $x \in \left[\frac{\pi}{4}, \pi\right]$, $\sin(x) \geq \cos(x)$, so

$|f(x) - g(x)| = |\sin(x) - \cos(x)| = \sin(x) - \cos(x)$.



Solution Part 2

Then,

$$\begin{aligned}
A &= \int_0^{\pi} |f(x) - g(x)|\, dx \\
&= \int_0^{\frac{\pi}{4}} (\cos(x) - \sin(x))\, dx + \int_{\frac{\pi}{4}}^{\pi} (\sin(x) - \cos(x))\, dx \\
&= [\sin(x) + \cos(x)]\Big|_0^{\frac{\pi}{4}} + [-\cos(x) - \sin(x)]\Big|_{\frac{\pi}{4}}^{\pi} \\
&= (\sqrt{2} - 1) + (1 + \sqrt{2}) \\
&= 2\sqrt{2}.
\end{aligned}$$

The area of the region is $2\sqrt{2}$ square units.

EXample

To find the area of region R between the graphs of $f(x) = \sin(x)$ and $g(x) = \cos(x)$ over the interval $\left[\frac{\pi}{2}, 2\pi\right]$, we can split the interval at the point where the curves intersect, $x = \frac{5\pi}{4}$.
For $x \in \left[\frac{\pi}{2}, \frac{5\pi}{4}\right]$, $\sin(x) \geq \cos(x)$, so $|f(x) - g(x)| = \sin(x) - \cos(x)$. On the other hand, for $x \in \left[\frac{5\pi}{4}, 2\pi\right]$, $\cos(x) \geq \sin(x)$, so $|f(x) - g(x)| = \cos(x) - \sin(x)$.

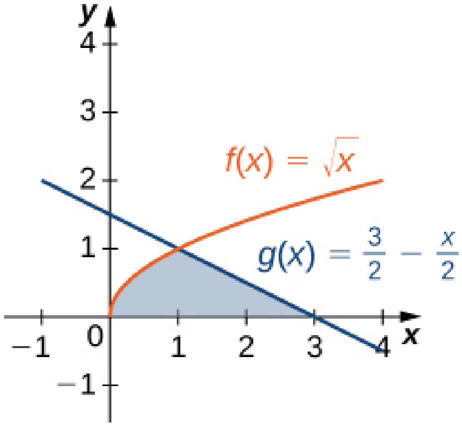
Solution Part 2

Then, we integrate each part separately and add the results.

$$
\begin{aligned}
A &= \int_{\frac{\pi}{2}}^{\frac{5\pi}{4}} (\sin(x) - \cos(x))\,dx + \int_{\frac{5\pi}{4}}^{2\pi} (\cos(x) - \sin(x))\,dx \\
&= [-\cos(x) - \sin(x)]\Big|_{\frac{\pi}{2}}^{\frac{5\pi}{4}} + [\sin(x) + \cos(x)]\Big|_{\frac{5\pi}{4}}^{2\pi} \\
&= (\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}) + 1 + 1 - (-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}) \\
&= 2(1 + \sqrt{2}).
\end{aligned}
$$

Therefore, the area of region R is $1 + \sqrt{2}$ square units.

Example



Consider the region depicted in the following figure. Find the area of R.

Answer: $\frac{5}{3}$ units 2

Hint: The two curves intersect at

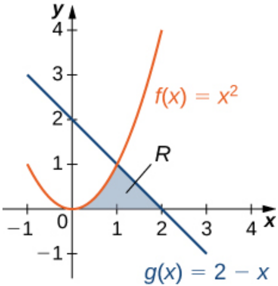
Finding the Area between Two Curves, Integrating along the y-axis

Let $u(y)$ and $v(y)$ be continuous functions such that $u(y) \geq v(y)$ for all $y \in [c, d]$. Let R denote the region bounded on the right by the graph of $u(y)$, on the left by the graph of $v(y)$, and above and below by the lines $y = d$ and $y = c$, respectively. Then, the area of R is given by

$$A = \int_c^d [u(y) - v(y)] \, dy.$$

Integrating with Respect to y

Let's revisit the example with the region shown in Figure 6, only this time let's integrate with respect to y. Let R be the region depicted below. Find the area of R by integrating with respect to y.



Solution

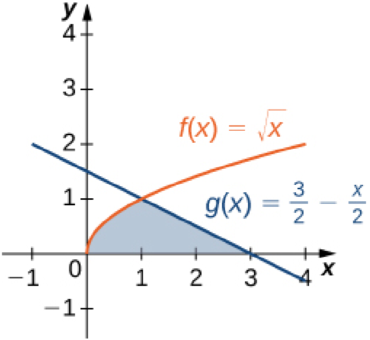
Solution: We must first express the graphs as functions of y. As we saw at the beginning of this section, the curve on the left can be represented by the function $x = v(y) = \sqrt{y}$, and the curve on the right can be represented by the function $x = u(y) = 2 - y$.
Now we have to determine the limits of integration. The region is bounded below by the x-axis, so the lower limit of integration is $y = 0$. The upper limit of integration is determined by the point where the two graphs intersect, which is the point $(1, 1)$, so the upper limit of integration is $y = 1$. Thus, we have $[c, d] = [0, 1]$.
Calculating the area of the region, we get

$$A = \int_c^d [u(y) - v(y)]\, dy = \int_0^1 [(2-y) - \sqrt{y}]\, dy = \left[2y - \frac{y^2}{2} - \frac{2}{3}y^{3/2}\right]\Bigg|_0^1 = \frac{5}{6}.$$

The area of the region is $\frac{5}{6}$ square units.

Example

Let's revisit the exercise associated with Figure 7, only this time, let's integrate with respect to y. Let be the region depicted in the following figure. Find the area of R by integrating with respect to y.



Consider the region depicted in the following figure. Find the area of R.

Answer: $\frac{5}{3}$ $units^2$

2.2 Determining Volumes by Slicing

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Outline

Learning Objectives

Determine the volume of a solid by integrating an area of a cross-section (the slicing method).

Find the volume of a solid of revolution using the disk method.

Find the volume of a solid of revolution with a cavity using the washer method.

Introduction to Volume Calculation

Just as area is the numerical measure of a two-dimensional region, volume is the numerical measure of a three-dimensional solid. Most of us have computed volumes of solids by using basic geometric formulas. For instance, the volume V of a rectangular solid, where l, w, and h represent its length, width, and height respectively, is given by:

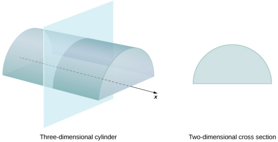
$$V = lwh$$

Similarly, formulas for the volumes of other common solids like the sphere ($V = \frac{4}{3}\pi r^3$), cone ($V = \frac{1}{3}\pi r^2 h$), and pyramid ($V = \frac{1}{3}Ah$) have been introduced. Although some of these formulas were derived using geometry alone, all of them can be obtained by using integration.

Volume of a cylinder

A cylinder is defined as any solid that can be generated by translating a plane region along a line perpendicular to the region, called the axis of the cylinder. To calculate the volume of a cylinder, we simply multiply the area of the cross-section A by the height h:

$$V = Ah$$

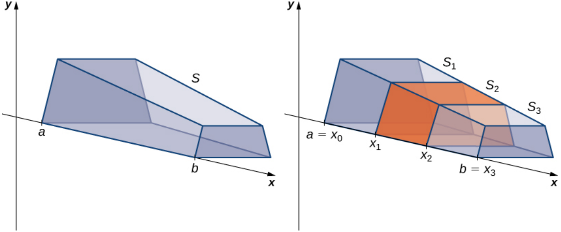


For a right circular cylinder (like a soup can), where r represents the radius of the circular base, this becomes:

$$V = \pi r^2 h$$

If a solid does not have a constant cross-section ?

We may not have a formula for its volume. In this case, we can use a definite integral to calculate the volume of the solid.



We divide S into slices perpendicular to the x-axis. Now let $P = \{x_0, x_1, \ldots, x_n\}$ be a regular partition of $[a, b]$, and for $i = 1, 2, \ldots, n$, let S_i represent the slice of the solid S stretching from x_{i-1} to x_i. Then the volume of slice S_i can be estimated by $V(S_i) \approx A(x_i^*)\Delta x$. Then we have

$$V(S) = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\Delta x = \int_a^b A(x)\, dx$$

Problem-Solving Strategy: Finding Volumes by the Slicing Method

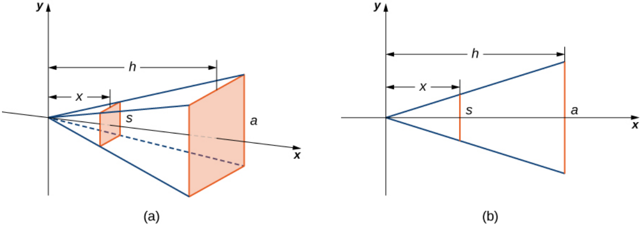
1. **Examine the solid and determine the shape of a cross-section of the solid.** It is often helpful to draw a picture if one is not provided.
2. **Determine a formula for the area of the cross-section.**
3. **Integrate the area expression over the appropriate interval to get the volume.**

Deriving the Formula for the Volume of a Pyramid

We know from geometry that the formula for the volume of a pyramid is $V = \frac{1}{3}Ah$. If the pyramid has a square base, this becomes $V = \frac{1}{3}a^2h$, where a denotes the length of one side of the base. We are going to use the slicing method to derive this formula.

Solution

- **Step 1:** We first want to determine the shape of a cross-section of the pyramid. Since the base is a square, the cross-sections are squares as well.



Deriving the Formula for the Volume of a Pyramid

- **Step 2:** Now we want to determine a formula for the area of one of these cross-sectional squares. Looking at Figure 4 (b), and using a proportion, since these are similar triangles, we have
$$\frac{s}{a} = \frac{x}{h} \text{ or } s = \frac{ax}{h}.$$
Therefore, the area of one of the cross-sectional squares is
$$A(x) = s^2 = \left(\frac{ax}{h}\right)^2.$$

- **Step 3:** Then we find the volume of the pyramid by integrating from 0 to h:
$$V = \int_0^h A(x)\,dx = \int_0^h \left(\frac{ax}{h}\right)^2 dx = \frac{a^2}{h^2}\int_0^h x^2\,dx$$
$$= \left[\frac{a^2}{h^2}\left(\frac{1}{3}x^3\right)\right]\Bigg|_0^h = \frac{1}{3}a^2 h.$$

Using the Slicing Method to Derive the Volume of a Circular Cone

Problem: Use the slicing method to derive the formula $V = \frac{1}{3}\pi r^2 h$ for the volume of a circular cone. **Hint:** Use similar triangles, as in the example above.

Using the Slicing Method to Derive the Volume of a Circular Cone

Problem: Use the slicing method to derive the formula $V = \frac{1}{3}\pi r^2 h$ for the volume of a circular cone. **Hint:** Use similar triangles, as in the example above.

Solution:

- **Step 1:** Consider a circular cone with height h and radius r. We want to determine the shape of a cross-section of the cone. Since the cross-sections are circles with radius varying from 0 to r as we move from the tip to the base, we have circular cross-sections.
- **Step 2:** Now, let's determine a formula for the area of one of these circular cross-sections. Using similar triangles, we can establish that the radius R of a cross-section at height x from the tip of the cone is given by the proportion $\frac{R}{r} = \frac{x}{h}$. Solving for R, we get $R = \frac{rx}{h}$. Therefore, the area $A(x)$ of a cross-section at height x is given by

$$A(x) = \pi R^2 = \pi \left(\frac{rx}{h}\right)^2 = \frac{\pi r^2 x^2}{h^2}$$

Using the Slicing Method to Derive the Volume of a Circular Cone part 2

- **Step 3:** To find the volume of the cone, we integrate the area function $A(x)$ over the interval $[0, h]$, which represents the height of the cone:

$$\begin{aligned} V &= \int_0^h A(x)\,dx = \int_0^h \frac{\pi r^2 x^2}{h^2}\,dx \\ &= \frac{\pi r^2}{h^2}\int_0^h x^2\,dx = \frac{\pi r^2}{h^2}\left[\frac{x^3}{3}\right]_0^h \\ &= \frac{\pi r^2}{h^2}\left(\frac{h^3}{3} - 0\right) = \frac{\pi r^2 h}{3}. \end{aligned}$$

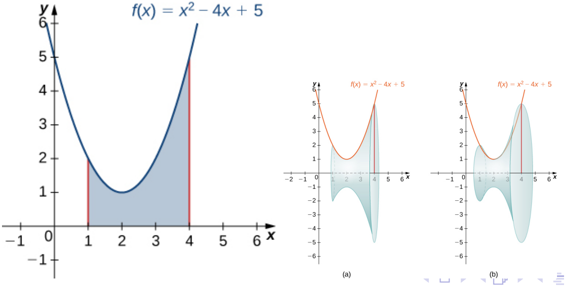
Thus, we have derived the formula for the volume of a circular cone: $V = \frac{1}{3}\pi r^2 h$.

Using the Slicing Method to Find the Volume of a Solid of Revolution

Problem: Use the slicing method to find the volume of the solid of revolution bounded by the graphs of $f(x) = x^2 - 4x + 5$, $x = 1$, and $x = 4$, and rotated about the x-axis.

Step 1: Sketch the graph of the quadratic function $f(x) = x^2 - 4x + 5$ over the interval $[1, 4]$. This region is used to produce a solid of revolution.

Step 2: Rotate the region around the x-axis to form the solid of revolution. Two views of the solid of revolution produced by revolving the region about the x-axis.



Find the Volume of a Solid of Revolution part 2

Step 3: Determine the shape of a cross-section of the solid of revolution.

- Since the solid was formed by revolving the region around the x-axis, the cross-sections are circles.
- The area of the cross-section is the area of a circle, and the radius of the circle is given by $f(x)$.

Step 4: Write the formula for the area of the cross-section.

$$A(x) = \pi[f(x)]^2 = \pi[(x^2 - 4x + 5)^2]$$

Step 5: Integrate the area function $A(x)$ over the interval $[1, 4]$ to find the volume.

$$V = \int_1^4 A(x)\, dx = \int_1^4 \pi[(x^2 - 4x + 5)^2]\, dx$$

Step 6: Evaluate the integral to find the volume.

$$V+ = \pi \left(\frac{x^5}{5} - 2x^4 + \frac{26x^3}{3} - 20x^2 + 25x \right)\Bigg|_1^4 = \frac{78}{5}\pi.$$

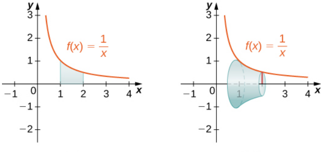
Using the Method of Slicing to Find Volume

Problem: Use the method of slicing to find the volume of the solid of revolution formed by revolving the region between the graph of the function $f(x) = \frac{1}{x}$ and the x-axis over the interval $[1, 2]$ around the x-axis.

Solution:

Step 1: Sketch the region to be rotated and identify the shape of the cross-sections.

- The region is between the graph of $f(x) = \frac{1}{x}$ and the x-axis over the interval $[1, 2]$.
- The cross-sections formed by revolving this region around the x-axis are disks.



(a) (b)

Method of Slicing to Find Volume Part 2

Step 2: Determine the formula for the area of a cross-section at height x.

- The radius of each disk is the function value $f(x) = \frac{1}{x}$.
- So, the area of a cross-section at height x is given by $A(x) = \pi[f(x)]^2 = \pi \left(\frac{1}{x}\right)^2 = \frac{\pi}{x^2}$.

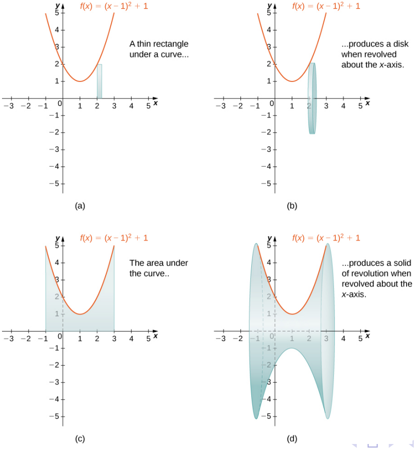
Step 3: Integrate the area function $A(x)$ over the interval $[1, 2]$ to find the volume.

$$V = \int_1^2 A(x)\,dx = \int_1^2 \frac{\pi}{x^2}\,dx = \pi \int_1^2 \frac{1}{x^2}\,dx$$
$$= \pi \left[-\frac{1}{x}\right]_1^2 = \pi \left(-\frac{1}{2} + 1\right) = \frac{\pi}{2}.$$

Therefore, the volume of the solid of revolution is $\frac{\pi}{2}$.

The Disk Method

When we use the slicing method with solids of revolution, it is often called the disk method because, for solids of revolution, the slices used to over-approximate the volume of the solid are disks.



The Disk Method

Let $f(x)$ be continuous and nonnegative. Define R as the region bounded above by the graph of $f(x)$, below by the x-axis, on the left by the line $x = a$, and on the right by the line $x = b$. Then, the volume of the solid of revolution formed by revolving R around the x-axis is given by

Formula

$$V = \int_a^b \pi[f(x)]^2 \, dx.$$

This formula represents the volume using the disk method.

Example: Volume Calculation using the Disk Method

The volume of the solid we have been studying is given by

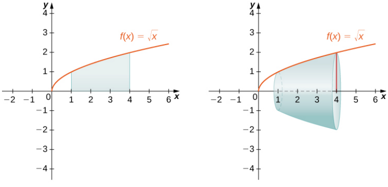
$$
\begin{aligned}
V &= \int_a^b \pi[f(x)]^2\,dx \\
&= \int_{-1}^3 \pi[(x-1)^2+1]^2\,dx \\
&= \pi\int_{-1}^3 [(x-1)^4+2(x-1)^2+1]^2\,dx \\
&= \pi\left[\frac{1}{5}(x-1)^5+\frac{2}{3}(x-1)^3+x\right]\Bigg|_{-1}^3 \\
&= \pi\left[\left(\frac{32}{5}+\frac{16}{3}+3\right)-\left(-\frac{32}{5}-\frac{16}{3}-1\right)\right] \\
&= \frac{412\pi}{15}\text{ units}^3.
\end{aligned}
$$

Using the Disk Method to Find Volume of a Solid of Revolution

Problem: Use the disk method to find the volume of the solid of revolution generated by rotating the region between the graph of $f(x) = \sqrt{x}$ and the x-axis over the interval $[1, 4]$ around the x-axis.

Solution:

1. The graphs of the function and the solid of revolution are shown in the following figure.



Using the Disk Method to Find Volume of a Solid of Revolution

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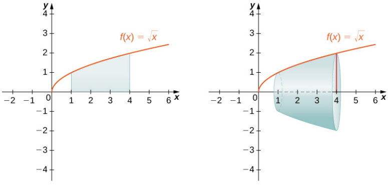


(a) (b)

$$V = \int_1^4 \pi[f(x)]^2\,dx = \int_1^4 \pi[\sqrt{x}]^2\,dx = \pi\int_1^4 x\,dx = \frac{\pi}{2}x^2\Big|_1^4 = \frac{15\pi}{2}\text{units}^3.$$

Using the Disk Method to Find Volume of a Solid of Revolution

Problem: Use the disk method to find the volume of the solid of revolution generated by rotating the region between the graph of $f(x) = \sqrt{4 - x}$ and the x-axis over the interval $[0, 4]$ around the x-axis.

Using the Disk Method to Find Volume of a Solid of Revolution

Problem: Use the disk method to find the volume of the solid of revolution generated by rotating the region between the graph of $f(x) = \sqrt{4-x}$ and the x-axis over the interval $[0, 4]$ around the x-axis.
Solution:

1. We have

$$V = \int_0^4 \pi [f(x)]^2 \, dx = \int_0^4 \pi [\sqrt{4-x}]^2 \, dx = \pi \int_0^4 (4-x) \, dx$$
$$= \pi \left[4x - \frac{x^2}{2} \right] \Bigg|_0^4 = \pi (16 - 8) = 8\pi \text{ units}^3.$$

The Disk Method for Solids of Revolution around the y-axis

Let $g(y)$ be continuous and nonnegative. Define Q as the region bounded on the right by the graph of $g(y)$, on the left by the y-axis, below by the line $y = c$, and above by the line $y = d$. Then, the volume of the solid of revolution formed by revolving Q around the y-axis is given by

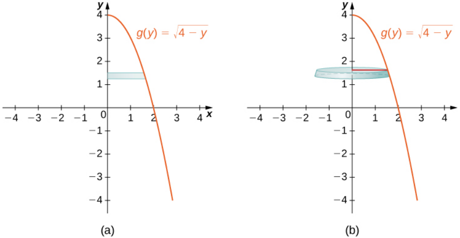
Formula

$$V = \int_c^d \pi[g(y)]^2\, dy.$$

Using the Disk Method to Find Volume of a Solid of Revolution 2

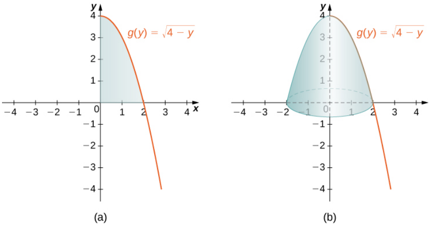
Problem: Let R be the region bounded by the graph of $g(y) = \sqrt{4-y}$ and the y-axis over the y-axis interval $[0, 4]$. Use the disk method to find the volume of the solid of revolution generated by rotating R around the y-axis.

Solution: This figure shows a thin rectangle between the curve of the function $g(y) = \sqrt{4-y}$ and the y-axis and the rectangle forms a representative disk after revolution around the y-axis



Solution Part 2

- The region to be revolved and the full solid of revolution are depicted in this Figure.



- To find the volume, we integrate with respect to y. We obtain

$$V = \int_0^4 \pi[g(y)]^2\,dy = \int_0^4 \pi[\sqrt{4-y}]^2\,dy = \pi\int_0^4 (4-y)\,dy$$

$$= \pi\left[4y - \frac{y^2}{2}\right]\Bigg|_0^4 = 8\pi \text{ units}^3.$$

Using the Disk Method to Find Volume of a Solid of Revolution

Problem: Use the disk method to find the volume of the solid of revolution generated by rotating the region between the graph of $g(y) = y$ and the y-axis over the interval $[1, 4]$ around the y-axis.

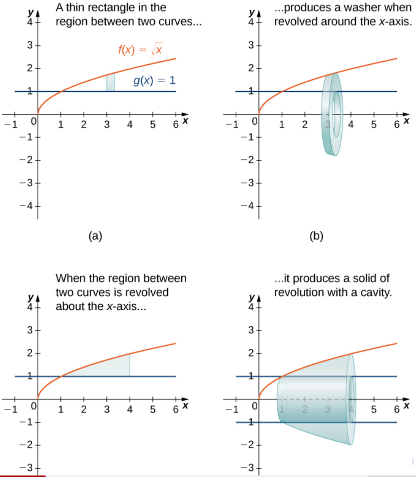
Solution:

1. To find the volume, we integrate with respect to y. We obtain

$$V = \int_1^4 \pi[g(y)]^2\,dy = \int_1^4 \pi[y]^2\,dy = \pi\int_1^4 y^2\,dy = \pi\left[\frac{1}{3}y^3\right]\Bigg|_1^4$$
$$= \pi\left(\frac{64}{3} - \frac{1}{3}\right) = \frac{63\pi}{3}$$
$$= 21\pi \text{units}^3.$$

The Washer Mathod

When the solid of revolution has a cavity in the middle, the slices used to approximate the volume are not disks, but washers (disks with holes in the center). For example, consider $f(x) = \sqrt{x}$ and $g(x) = 1$ over the interval $[1, 4]$.



The Washer Method

The cross-sectional area, then, is the area of the outer circle minus the area of the inner circle. In this case,

$$\begin{aligned} A(x) &= \pi[(\sqrt{x})^2 - (1)^2] \\ &= \pi(x-1). \end{aligned}$$

Then the volume of the solid is

$$\begin{aligned} V = \int_a^b A(x)\,dx &= \int_1^4 \pi(x-1)\,dx \\ &= \pi\left[\frac{x^2}{2} - x\right]\Bigg|_1^4 \\ &= \frac{9}{2}\pi \text{ units}^3. \end{aligned}$$

Generalizing this process gives the washer method.

The Washer Method

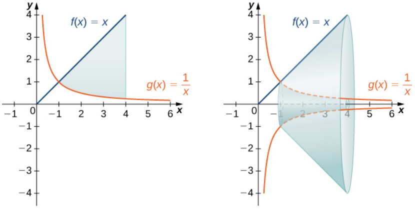
Suppose $f(x)$ and $g(x)$ are continuous, nonnegative functions such that $f(x) \geq g(x)$ over $[a, b]$. Let R denote the region bounded above by the graph of $f(x)$, below by the graph of $g(x)$, on the left by the line $x = a$, and on the right by the line $x = b$. Then, the volume of the solid of revolution formed by revolving R around the x-axis is given by

Formula

$$V = \int_a^b \pi[(f(x))^2 - (g(x))^2]\,dx.$$

Using the Washer Method

Problem: Find the volume of a solid of revolution formed by revolving the region bounded above by the graph of $f(x) = x$ and below by the graph of $g(x) = \frac{1}{x}$ over the interval $[1, 4]$ around the x-axis.



(a) (b)

Solution:

1. The volume is given by

$$V = \int_a^b \pi[(f(x))^2 - (g(x))^2]\,dx = \pi \int_1^4 [x^2 - (\frac{1}{x})^2]\,dx$$

$$= \pi \int_1^4 [x^2 - \frac{1}{x^2}]\,dx = \pi \left[\frac{x^3}{3} + \frac{1}{x}\right]\Bigg|_1^4 \frac{81\pi}{4} \text{ units}^3.$$

Using the Washer Method

Problem: Find the volume of a solid of revolution formed by revolving the region bounded by the graphs of $f(x) = \sqrt{x}$ and $g(x) = \frac{1}{x}$ over the interval $[1, 3]$ around the x-axis.

Using the Washer Method

Problem: Find the volume of a solid of revolution formed by revolving the region bounded by the graphs of $f(x) = \sqrt{x}$ and $g(x) = \frac{1}{x}$ over the interval $[1, 3]$ around the x-axis.

Solution:

1. Determine which function forms the upper bound and which forms the lower bound by graphing them.
2. Once you have identified the bounds, use the Washer Method to find the volume:

$$V = \int_a^b \pi[(f(x))^2 - (g(x))^2]\, dx$$

3. Calculate the integral over the interval $[1, 3]$.

$$V = \int_1^3 \pi[(\sqrt{x})^2 - (\frac{1}{x})^2]\, dx = \pi \int_1^3 [x - \frac{1}{x^2}]\, dx = \pi \left[\frac{x^2}{2} + \frac{1}{x}\right]\Bigg|_1^3 = \frac{10\pi}{3} \text{ units}^3.$$

Rule: The Washer Method for Solids of Revolution around the y-axis

Suppose $u(y)$ and $v(y)$ are continuous, nonnegative functions such that $v(y) \leq u(y)$ for $y \in [c, d]$. Let Q denote the region bounded on the right by the graph of $u(y)$, on the left by the graph of $v(y)$, below by the line $y = c$, and above by the line $y = d$. Then, the volume of the solid of revolution formed by revolving Q around the y-axis is given by

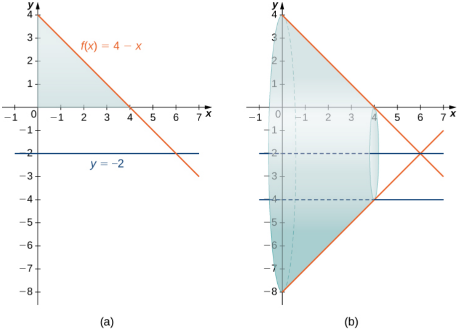
Formula

$$V = \int_c^d \pi \left[(u(y))^2 - (v(y))^2 \right] dy.$$

The Washer Method with a Different Axis of Revolution

Find the volume of a solid of revolution formed by revolving the region bounded above by $f(x) = 4 - x$ and below by the x-axis over the interval $[0, 4]$ around the line $y = -2$.

Solution: The graph of the region and the solid of revolution are shown in the following figure.



There is a solid formed by rotating the shaded region from the first graph around the line $y = -2$. There is a hollow cylinder inside of the solid represented by the lines $y = -2$ and $y = -4$.

Solution part 2

We can't apply the volume formula to this problem directly because the axis of revolution is not one of the coordinate axes. However, we still know that the area of the cross-section is the area of the outer circle less the area of the inner circle. Looking at the graph of the function, we see the radius of the outer circle is given by $f(x) + 2$, which simplifies to

$$f(x) + 2 = (4 - x) + 2 = 6 - x.$$

The radius of the inner circle is $g(x) = 2$. Therefore, we have

$$V = \int_0^4 \pi[(6-x)^2 - (2)^2]dx = \pi \int_0^4 (x^2 - 12x + 32)dx = \frac{160\pi}{3} \text{ units}^3.$$

Volume of a Solid of Revolution

Find the volume of a solid of revolution formed by revolving the region bounded above by the graph of $f(x) = x + 2$ and below by the x-axis over the interval $[0, 3]$ around the line $y = -1$.

Solution: Let us use the method of washers.

- **Step 1:** The radius of the outer circle (R) is the distance from the axis of revolution $(y = -1)$ to the outer function $(f(x) = x + 2)$. So, $R = f(x) + 1 = (x + 2) + 1 = x + 3$.

  The radius of the inner circle (r) is the distance from the axis of revolution to the x-axis, which is simply the constant 1.

- **Step 2:** The region of revolution is bounded by $x = 0$ and $x = 3$.

Solution part 2

Step 3: Now, we can set up the integral to compute the volume using the washer method:

$$V = \pi \int_0^3 (R^2 - r^2)\,dx.$$

Substituting $R = x + 3$ and $r = 1$, we have:

$$V = \pi \int_0^3 ((x+3)^2 - 1)\,dx.$$

Step 4: Evaluate the integral:

$$V = \pi \int_0^3 (x^2 + 6x + 8)\,dx = \pi \left[\frac{x^3}{3} + 3x^2 + 8x \right]_0^3$$

$$V = \pi \left(\frac{27}{3} + 27 + 24 \right) = \pi\,(9 + 27 + 24) = \pi \times 60 = 60\pi \text{ units}^3.$$

Key Concepts

- Definite integrals can be used to find the volumes of solids. Using the slicing method, we can find a volume by integrating the cross-sectional area.
- For solids of revolution, the volume slices are often disks and the cross-sections are circles. The method of disks involves applying the method of slicing in the particular case in which the cross-sections are circles, and using the formula for the area of a circle.
- If a solid of revolution has a cavity in the center, the volume slices are washers. With the method of washers, the area of the inner circle is subtracted from the area of the outer circle before integrating.

Key Equations

Disk Method along the x-axis:

$$V = \int_a^b \pi[f(x)]^2\,dx$$

Disk Method along the y-axis:

$$V = \int_c^d \pi[g(y)]^2\,dy$$

Washer Method:

$$V = \int_a^b \pi[(f(x))^2 - (g(x))^2]\,dx$$

2.3 Volumes of Revolution: Cylindrical Shells

Math 1700

University of Manitoba

2024

Outline

Learning Objectives

Objective 1

Calculate the volume of a solid of revolution by using the method of cylindrical shells.

Objective 2

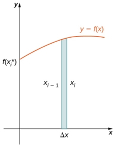
Compare the different methods for calculating the volume of a solid of revolution.

The Method of Cylindrical Shells

Again, we are working with a solid of revolution. As before, we define a region R, bounded above by the graph of a function y=f(x), below by the x-axis, and on the left and right by the lines x=a and x=b.

Figure 2/image.png

To calculate the volume of this shell, consider the figure below.



The volume of the shell is the product of the cross-sectional area and the height. Cross-sections are annuli with outer radius x_i and inner radius x_{i-1}, giving an area of $\pi x_i^2 - \pi x_{i-1}^2$. The height is $f(x_i^*)$.

$$\begin{aligned} V_{\text{shell}} &= f(x_i^*)(\pi x_i^2 - \pi x_{i-1}^2) \\ &= \pi f(x_i^*)(x_i^2 - x_{i-1}^2) \\ &= \pi f(x_i^*)(x_i + x_{i-1})(x_i - x_{i-1}) \\ &= 2\pi f(x_i^*)\left(\frac{x_i + x_{i-1}}{2}\right)(x_i - x_{i-1}). \end{aligned}$$

Note that $x_i - x_{i-1} = \Delta x$, so we have

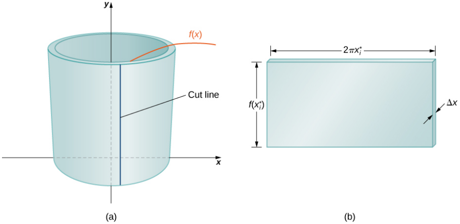
$$V_{\text{shell}} = 2\pi f(x_i^*)\left(\frac{x_i + x_{i-1}}{2}\right)\Delta x.$$

Furthermore, $\frac{x_i+x_{i-1}}{2}$ is both the midpoint of the interval $[x_{i-1}, x_i]$ and the average radius of the shell, and we can approximate this by x_i^*. We then have

$$V_{\text{shell}} \approx 2\pi f(x_i^*)x_i^*\Delta x.$$

Another approach

Another way to think of this is to think of making a vertical cut in the shell and then opening it up to form a flat plate.



Then

$$V_{\text{shell}} \approx 2\pi f(x_i^*)x_i^*\Delta x.$$

The Method of Cylindrical Shells

Let $f(x)$ be continuous and nonnegative. Define R as the region bounded above by the graph of $f(x)$, below by the x-axis, on the left by the line $x = a$, and on the right by the line $x = b$. Then the volume of the solid of revolution formed by revolving R around the y-axis is given by

General formula

$$V = \int_a^b (2\pi x f(x))\, dx.$$

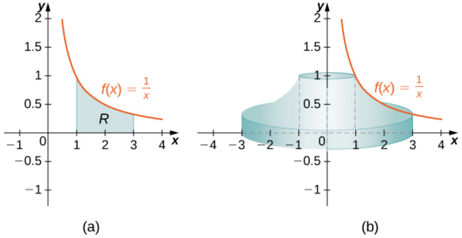
This formula represents the method of cylindrical shells, which is used to find the volume of a solid of revolution. It involves integrating the product of the circumference of a cylindrical shell (given by $2\pi x$) and the height of the shell (given by $f(x)$) over the interval $[a, b]$ along the x-axis.

The Method of Cylindrical Shells 1

Define R as the region bounded above by the graph of $f(x) = \frac{1}{x}$ and below by the x-axis over the interval $[1, 3]$. Find the volume of the solid of revolution formed by revolving R around the y-axis.

Solution

Solution: First we must graph the region R and the associated solid of revolution, as shown in the following figure.



Then the volume of the solid is given by

$$V = \int_a^b (2\pi x f(x))\,dx = \int_1^3 \left(2\pi x\left(\tfrac{1}{x}\right)\right)\,dx$$
$$= \int_1^3 2\pi\,dx = 2\pi x\Big|_1^3 = 4\pi \text{ units}^3.$$

Volume of a Solid of Revolution

Problem Statement:
Define R as the region bounded above by the graph of $f(x) = x^2$ and below by the x-axis over the interval $[1, 2]$. Find the volume of the solid of revolution formed by revolving R around the y-axis.
Solution: The volume of the solid of revolution is given by

$$V = \int_a^b 2\pi x f(x)\, dx.$$

Substituting $f(x) = x^2$ and $a = 1, b = 2$, we have

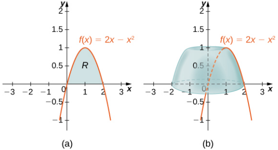
$$V = \int_1^2 2\pi x(x^2)\, dx.$$

This integrates to

$$V = \left[\frac{2\pi}{4}x^4\right]_1^2 = \frac{2\pi}{4}(2^4 - 1^4) = \frac{15\pi}{2} \text{ units}^3.$$

Volume of a Solid of Revolution

Problem Statement:
Define R as the region bounded above by the graph of $f(x) = 2x - x^2$ and below by the x-axis over the interval $[0, 2]$. Find the volume of the solid of revolution formed by revolving R around the y-axis.



Solution: The volume of the solid of revolution is given by

$$V = \int_a^b 2\pi x f(x)\, dx.$$

Substituting $f(x) = x^2$ and $a = 0, b = 2$, we have

$$V = \int_0^2 2\pi x(2x - x^2)\, dx = 2\pi \left[\frac{2}{3}x^3 - \frac{x^4}{4}\right]_0^2 = \frac{8\pi}{3} \text{ units}^3.$$

The Method of Cylindrical Shells for Solids of Revolution around the x-axis

Let $g(y)$ be continuous and nonnegative. Define Q as the region bounded on the right by the graph of $g(y)$, on the left by the y-axis, below by the line $y = c$, and above by the line $y = d$. Then, the volume of the solid of revolution formed by revolving Q around the x-axis is given by

Formula

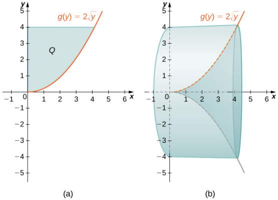
$$V = \int_c^d (2\pi y g(y))\, dy.$$

he Method of Cylindrical Shells for a Solid Revolved around the x-axis

Define Q as the region bounded on the right by the graph of $g(y) = 2\sqrt{y}$ and on the left by the y-axis for $y \in [0, 4]$. Find the volume of the solid of revolution formed by revolving Q around the x-axis.

he Method of Cylindrical Shells for a Solid Revolved around the x-axis

Define Q as the region bounded on the right by the graph of $g(y) = 2\sqrt{y}$ and on the left by the y-axis for $y \in [0, 4]$. Find the volume of the solid of revolution formed by revolving Q around the x-axis.



Label the shaded region Q. Then the volume of the solid is given by

$$V = \int_c^d (2\pi y g(y))\, dy = \int_0^4 (2\pi y(2\sqrt{y}))\, dy = 4\pi \int_0^4 y^{3/2}\, dy$$

$$= \frac{4\pi}{5}\left[\frac{2y^{5/2}}{5}\right]\Bigg|^4 = \frac{256\pi}{5} \text{ units}^3.$$

Solution

Problem Statement

Define Q as the region bounded on the right by the graph of $g(y) = \frac{3}{y}$ and on the left by the y-axis for $y \in [1, 3]$. Find the volume of the solid of revolution formed by revolving Q around the x-axis.

Solution

Problem Statement

Define Q as the region bounded on the right by the graph of $g(y) = \frac{3}{y}$ and on the left by the y-axis for $y \in [1, 3]$. Find the volume of the solid of revolution formed by revolving Q around the x-axis.

To find the volume of the solid of revolution, we use the method of cylindrical shells. The volume V is given by the integral

$$V = \int_c^d (2\pi y g(y))\, dy.$$

Here, $c = 1$ and $d = 3$, and $g(y) = \frac{3}{y}$. Thus,

$$V = \int_1^3 (2\pi y \cdot \frac{3}{y})\, dy = 6\pi \int_1^3 dy = 6\pi \cdot (3 - 1) = 12\pi \text{ units}^3.$$

Therefore, the volume of the solid of revolution is $\boxed{12\pi \text{ units}^3}$.

Graph of a function around a line other than x=0 and y=0

We look at a solid of revolution for which the graph of a function is revolved around a line other than one of the two coordinate axes. Suppose, for example, that we rotate the region around the line x=-k, where k is some positive constant.

The formula for the volume of the shell is

Formula when $x = -k$

$$V = \int_a^b (2\pi(x + k)f(x))\,dx.$$

We could also rotate the region around other horizontal or vertical lines, such as a vertical line in the right half plane. In each case, the volume formula must be adjusted accordingly. Specifically, the x-term in the integral must be replaced with an expression representing the radius of a shell.

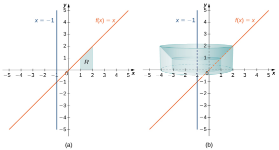
A Region of Revolution Revolved around a Line

Define R as the region bounded above by the graph of $f(x) = x$ and below by the x-axis over the interval $[1, 2]$. Find the volume of the solid of revolution formed by revolving R around the line $x = -1$.

A Region of Revolution Revolved around a Line

Define R as the region bounded above by the graph of $f(x) = x$ and below by the x-axis over the interval $[1, 2]$. Find the volume of the solid of revolution formed by revolving R around the line $x = -1$.

Solution The Figure is



In that case, the radius of a shell is given by $x + 1$. Then,

$$V = \int_1^2 (2\pi(x+1)x)\,dx = 2\pi \int_1^2 (x^2 + x)\,dx = 2\pi \left[\frac{x^3}{3} + \frac{x^2}{2}\right]\Bigg|_1^2 = \frac{23\pi}{3} \text{ units}^3.$$

Solid of Revolution around a Line

Define R as the region bounded above by the graph of $f(x) = x^2$ and below by the x-axis over the interval $[0, 1]$. Find the volume of the solid of revolution formed by revolving R around the line $x = -2$.

Solid of Revolution around a Line

Define R as the region bounded above by the graph of $f(x) = x^2$ and below by the x-axis over the interval $[0, 1]$. Find the volume of the solid of revolution formed by revolving R around the line $x = -2$.

Solution To find the volume of the solid of revolution, we use the method of cylindrical shells. Note that the radius of a shell is given by $x + 2$. Then the volume V is given by the integral

$$V = \int_0^1 (2\pi(x+2)f(x))\,dx.$$

$$V = \int_0^1 (2\pi(x+2)x^2)\,dx = 2\pi\int_0^1 (x^3 + 2x^2)\,dx$$

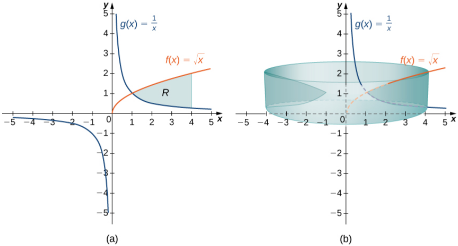
$$= 2\pi\left[\frac{x^4}{4} + \frac{2x^3}{3}\right]\Big|_0^1 = \frac{11\pi}{6} \text{ units}^3.$$

A Region of Revolution Bounded by Two Functions

Problem Statement

Define R as the region bounded above by the graph of the function $f(x) = \sqrt{x}$ and below by the graph of the function $g(x) = \frac{1}{x}$ over the interval $[1, 4]$. Find the volume of the solid of revolution generated by revolving R around the y-axis.

Solution: To find the volume of the solid of revolution, we use the method of cylindrical shells. Note that the axis of revolution is the y-axis, so the radius of a shell is simply x.



Solution part 2

However, the height of a shell is given by $f(x) - g(x)$, so we need to adjust the $f(x)$ term of the integrand. Then the volume V is given by the integral

$$V = \int_1^4 (2\pi x(f(x) - g(x)))\, dx.$$

$$\begin{aligned} V &= \int_1^4 \left(2\pi x \left(\sqrt{x} - \frac{1}{x}\right)\right) dx = 2\pi \int_1^4 (x^{3/2} - 1)\, dx \\ &= 2\pi \left[\frac{2x^{5/2}}{5} - x\right]\Bigg|_1^4 \\ &= \frac{94\pi}{5} \text{ units}^3. \end{aligned}$$

Therefore, the volume of the solid of revolution is $\boxed{\frac{94\pi}{5} \text{ units}^3}$.

Region Bounded by Two Functions

Define R as the region bounded above by the graph of $f(x) = x$ and below by the graph of $g(x) = x^2$ over the interval $[0, 1]$. Find the volume of the solid of revolution formed by revolving R around the y-axis.

Region Bounded by Two Functions

Define R as the region bounded above by the graph of $f(x) = x$ and below by the graph of $g(x) = x^2$ over the interval $[0, 1]$. Find the volume of the solid of revolution formed by revolving R around the y-axis.

Solution: Note that the axis of revolution is the y-axis, so the radius of a shell is simply x. However, the height of a shell is given by $f(x) - g(x)$, so we need to adjust the $f(x)$ term of the integrand. Then the volume V is given by the integral

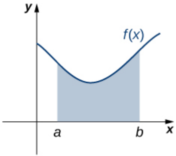
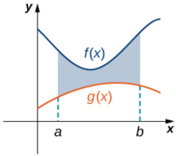
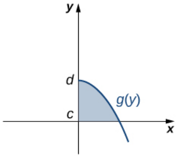
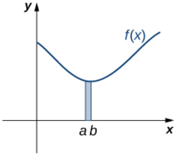
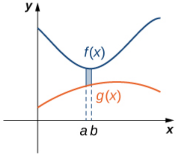
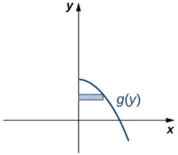
$$V = \int_0^1 (2\pi x(f(x) - g(x)))\, dx.$$

$$V = \int_0^1 \left(2\pi x\left(x - x^2\right)\right)\, dx = 2\pi \int_0^1 (x^2 - x^3)\, dx$$

$$= 2\pi \left[\frac{x^3}{3} - \frac{x^4}{4}\right]\Bigg|_0^1 = \frac{\pi}{6}\,\text{units}^3.$$

Which Method Should We Use?

Comparing the Methods for Finding the Volume of a Solid Revolution around the x-axis

Compare	Disk Method	Washer Method	Shell Method
Volume formula	$V = \int_a^b \pi[f(x)]^2\,dx$	$V = \int_a^b \pi\left[(f(x))^2 - (g(x))^2\right]dx$	$V = \int_c^d 2\pi y\, g(y)\,dy$
Solid	No cavity in the center	Cavity in the center	With or without a cavity in the center
Interval to partition	$[a, b]$ on x-axis	$[a, b]$ on x-axis	$[c, d]$ on y-axis
Rectangle	Vertical	Vertical	Horizontal
Typical region			
Typical element			

Select the best Method

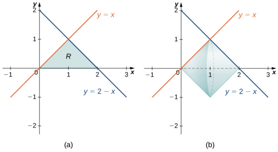
For each of the following problems, select the best method to find the volume of a solid of revolution generated by revolving the given region around the x-axis, and set up the integral to find the volume (do not evaluate the integral).

(a) The region bounded by the graphs of $y = x$, $y = 2 - x$, and the x-axis.

(b) The region bounded by the graphs of $y = 4x - x^2$ and the x-axis.

Solution Problem (a)

First, sketch the region and the solid of the revolution as shown



If we want to integrate concerning x, we would have to break the integral into two pieces because we have different functions bounding the region over $[0, 1]$ and $[1, 2]$. In this case, using the disk method, we would have

$$V = \int_0^1 (\pi x^2)\, dx + \int_1^2 (\pi(2-x)^2)\, dx$$

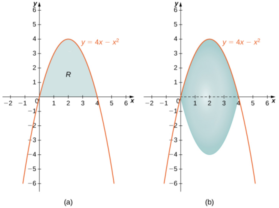
If we used the shell method instead,

$$V = \int_0^1 (2\pi y\,[(2-y)-y])dy = \int_0^1 (2\pi y\,[2-2y])dy.$$

Neither of these integrals is particularly onerous, but since the shell method requires only one integral, and the integrand requires less simplification, we should probably go with the shell method in this case.

Solution Problem (b)

First, sketch the region and the solid of revolution as shown.



Looking at the region, it would be problematic to define a horizontal rectangle; the region is bounded on the left and right by the same function. Therefore, we can dismiss the method of shells. The solid has no cavity in the middle, so we can use the method of disks. Then

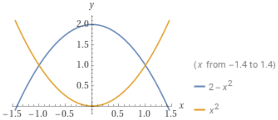
$$V = \int_0^4 \pi(4x - x^2)^2 \, dx.$$

Selecting the Best Method

For the region bounded by the graphs of $y = 2 - x^2$ and $y = x^2$ around the x-axis.

Selecting the Best Method

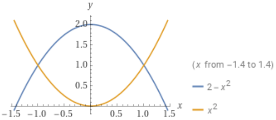
For the region bounded by the graphs of $y = 2 - x^2$ and $y = x^2$ around the x-axis. Sketch the region and use the table below to decide which method works best.



Shape of Region	Best Method
No cavity in the center	Washer method
Cavity in the center	Disk method
With or without cavity in the center	Shell method

Selecting the Best Method

For the region bounded by the graphs of $y = 2 - x^2$ and $y = x^2$ around the x-axis. Sketch the region and use the table below to decide which method works best.



Shape of Region	Best Method
No cavity in the center	Washer method
Cavity in the center	Disk method
With or without cavity in the center	Shell method

Solution The best method is the washer method.

$$V = \int_{-1}^{1} \pi \left[(2 - x^2)^2 - (x^2)^2\right] \, dx$$

Key Equations

Method of Cylindrical Shells

The method of cylindrical shells is another method for using a definite integral to calculate the volume of a solid of revolution. This method is sometimes preferable to either the method of disks or the method of washers because we integrate with respect to the other variable. In some cases, one integral is substantially more complicated than the other. The geometry of the functions and the difficulty of the integration are the main factors in deciding which integration method to use.

Key Equations

Method of Cylindrical Shells

The method of cylindrical shells is another method for using a definite integral to calculate the volume of a solid of revolution. This method is sometimes preferable to either the method of disks or the method of washers because we integrate with respect to the other variable. In some cases, one integral is substantially more complicated than the other. The geometry of the functions and the difficulty of the integration are the main factors in deciding which integration method to use.

Method of Cylindrical Shells

$$V = \int_a^b (2\pi x f(x))\, dx$$

2.4 Arc Length of a Curve and Surface Area

Math 1700

University of Manitoba

September 25, 2025

Outline

1. Arc Length of the Curve y=f(x)
2. Arc Length of the Curve $x = g(y)$
3. Area of a Surface of Revolution

Learning Objectives

- Determine the length of a curve with equation $y = f(x)$ between the two given points.
- Determine the length of a curve with equation $x = g(y)$ between the two given points.
- Find the surface area of a surface of revolution.

Motivation

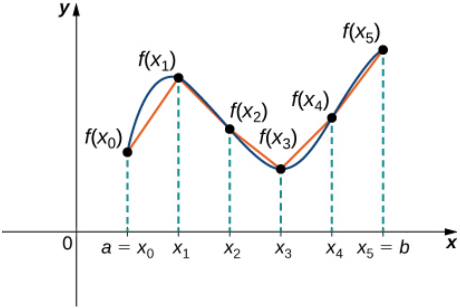
In this section, we use definite integrals to find the arc length of curves, which has various real-world applications such as determining the distance traveled by a rocket along a parabolic path or the driving distance along a road represented by a curve on a map.

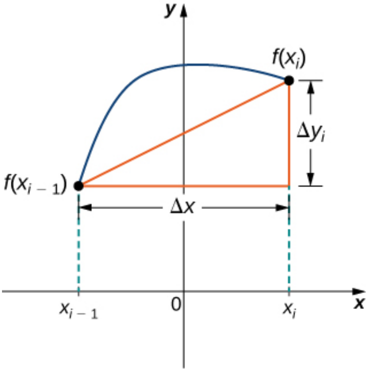
We start by calculating arc length for curves defined as functions of x, then similarly for curves defined as functions of y.

Finally, we extend these techniques to find the surface area of a surface of revolution.

Length of a Curve Approximation

Let $f(x)$ be a smooth function defined over $[a, b]$. We want to calculate the length of the curve from the point $(a, f(a))$ to the point $(b, f(b))$. We start by using line segments to approximate the length of the curve. For $i = 0, 1, 2, \ldots, n$, let $P = \{x_i\}$ be a regular partition of $[a, b]$.





Arc Length for $y = f(x)$

Let $f(x)$ be a smooth function over the interval $[a, b]$. Then the arc length of the portion of the graph of $f(x)$ from the point $(a, f(a))$ to the point $(b, f(b))$ is given by

Formula

$$\text{Arc Length} = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx.$$

Calculating the Arc Length of a Function of x

Let $f(x) = 2x^{3/2}$. Calculate the arc length of the graph of $f(x)$ over the interval $[0, 1]$.

Solution: We have $f'(x) = 3x^{1/2}$, so $[f'(x)]^2 = 9x$. Then, the arc length is

$$\text{Arc Length} = \int_0^1 \sqrt{1 + [f'(x)]^2}\, dx.$$

Substitute $u = 1 + 9x$. Then, $du = 9dx$. When $x = 0$, then $u = 1$, and when $x = 1$, then $u = 10$. Thus,

$$\begin{aligned}\text{Arc Length} &= \int_0^1 \sqrt{1 + 9x}\, dx \\ &= \frac{1}{9}\int_0^1 \sqrt{1 + 9x} \cdot 9\, dx = \frac{1}{9}\int_1^{10} \sqrt{u}\, du \\ &= \frac{1}{9} \cdot \frac{2}{3} u^{3/2}\Big|_1^{10} = \frac{2}{27}\left[10\sqrt{10} - 1\right] \text{ units.}\end{aligned}$$

Calculating the Arc Length of a Function of x

Let $f(x) = \frac{4}{3}x^{3/2}$. Calculate the arc length of the graph of $f(x)$ over the interval $[0, 1]$.

Solution: We have $f'(x) = 2x^{1/2}$, so $[f'(x)]^2 = 4x$. Then, the arc length is

$$\text{Arc Length} = \int_0^1 \sqrt{1 + [f'(x)]^2}\, dx.$$

Substitute $u = 1 + 4x$. Then, $du = 4dx$. When $x = 0$, then $u = 1$, and when $x = 1$, then $u = 5$. Thus,

$$\begin{aligned}\text{Arc Length} &= \int_0^1 \sqrt{1+4x}\, dx = \frac{1}{4}\int_0^1 \sqrt{1+4x} \cdot 4\, dx = \frac{1}{4}\int_1^5 \sqrt{u}\, du \\ &= \frac{1}{4} \cdot \frac{2}{3} u^{3/2}\Big|_1^5 = \frac{1}{6}(5\sqrt{5} - 1).\end{aligned}$$

Setting Up the Integral for the Arc Length of a Function of x

Let $f(x) = x^2$. Set up the integral for the arc length of the graph of $f(x)$ over the interval $[1, 3]$.

Solution: We have $f'(x) = 2x$, so $[f'(x)]^2 = 4x^2$. Then the arc length is given by

$$\text{Arc Length} = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx = \int_1^3 \sqrt{1 + 4x^2}\, dx.$$

Setting Up the Integral for the Arc Length of a Function

Let $f(x) = \ln(x)$. Set up the integral for the arc length of the graph of $f(x)$ over the interval $[1, e]$.

Solution: We have $f'(x) = \frac{1}{x}$, so $[f'(x)]^2 = \frac{1}{x^2}$. Then the arc length is given by

$$\text{Arc Length} = \int_1^e \sqrt{1 + \left[\frac{1}{x}\right]^2}\, dx = \int_1^e \sqrt{1 + \frac{1}{x^2}}\, dx.$$

Setting Up the Integral for the Arc Length of a Function

Let $f(x) = \ln(x)$. Set up the integral for the arc length of the graph of $f(x)$ over the interval $[1, e]$.

Solution: We have $f'(x) = \frac{1}{x}$, so $[f'(x)]^2 = \frac{1}{x^2}$. Then the arc length is given by

$$\text{Arc Length} = \int_1^e \sqrt{1 + \left[\frac{1}{x}\right]^2}\, dx = \int_1^e \sqrt{1 + \frac{1}{x^2}}\, dx.$$

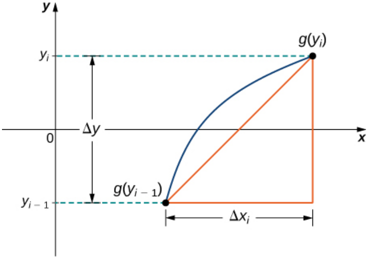
Substituting $u = \frac{1}{x}$, we have $du = -\frac{1}{x^2} dx$. When $x = 1$, then $u = 1$, and when $x = e$, then $u = \frac{1}{e}$. Thus,

$$\begin{aligned}\text{Arc Length} &= \int_1^{\frac{1}{e}} \sqrt{1 + u^2} \cdot \left(-\frac{1}{u^2}\right) du \\ &= -\int_1^{\frac{1}{e}} \sqrt{1 + u^2} \cdot \frac{1}{u^2}\, du = -\int_1^{\frac{1}{e}} \frac{\sqrt{1 + u^2}}{u^2}\, du.\end{aligned}$$

This integral might not have a straightforward antiderivative, but setting up the integral is the first step in finding the arc length.

Arc Length of the Curve $x = g(y)$

We have just seen how to approximate the length of a curve with line segments. If we want to find the arc length of the graph of a function of y, we can repeat the same process, except we partition the y-axis instead of the x-axis.



Arc Length for $x = g(y)$

Let $g(y)$ be a smooth function over an interval $[c, d]$. Then, the arc length of the graph of $g(y)$ from the point $(c, g(c))$ to the point $(d, g(d))$ is given by

Formula

$$\text{Arc Length} = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy.$$

Setting Up the Integral for Arc Length

Let $g(y) = 3y^3$. Set up the integral for the arc length of the graph of $g(y)$ over the interval $[1, 2]$.

Solution: We have $g'(y) = 9y^2$, so $[g'(y)]^2 = 81y^4$. Then the arc length is given by

$$\text{Arc Length} = \int_1^2 \sqrt{1 + [g'(y)]^2}\, dy = \int_1^2 \sqrt{1 + 81y^4}\, dy.$$

This integral may not have a straightforward antiderivative, but we can set up the integral as follows.

$$\text{Arc Length} = \int_1^2 \sqrt{1 + 81y^4}\, dy.$$

We can then proceed with numerical methods or other techniques to approximate the value of this integral.

Setting Up the Integral for Arc Length

Let $g(y) = \frac{1}{y}$. Set up the integral for the arc length of the graph of $g(y)$ over the interval $[1, 4]$.

Solution: We have $g'(y) = -\frac{1}{y^2}$, so $[g'(y)]^2 = \frac{1}{y^4}$. Then the arc length is given by

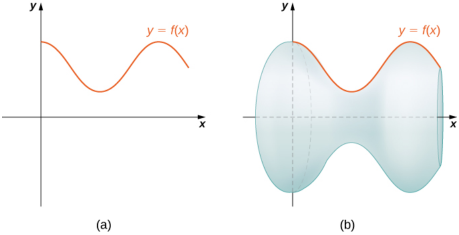
$$\text{Arc Length} = \int_1^4 \sqrt{1 + [g'(y)]^2}\, dy = \int_1^4 \sqrt{1 + \frac{1}{y^4}}\, dy.$$

Answer:

$$\text{Arc Length} = \int_1^4 \sqrt{1 + \frac{1}{y^4}}\, dy.$$

Area of a Surface of Revolution

Let $f(x)$ be a nonnegative smooth function over the interval $[a, b]$. We wish to find the surface area of the surface of revolution created by revolving the graph of $y = f(x)$ around the x-axis as shown in the following figure.



Surface Area of a Surface of Revolution

- Let $f(x)$ be a smooth function over the interval $[a, b]$.
- If $f(x) \geq 0$ for $x \in [a, b]$, then the area of the surface obtained by revolving the graph of $f(x)$ around the x-axis is given by

Formula

$$\text{Surface Area} = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\, dx.$$

This formula represents the surface area of the solid formed by rotating the curve $y = f(x)$ about the x-axis.

Surface Area of a Surface of Revolution

- Let $g(y)$ be a smooth function over the interval $[c, d]$.
- If $g(y) \geq 0$ for $y \in [c, d]$, then the area of the surface obtained by revolving the curve $x = g(y)$ around the y-axis is given by

Formula

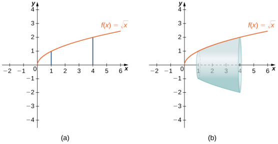
$$\text{Surface Area} = \int_c^d 2\pi g(y)\sqrt{1 + (g'(y))^2}\, dy.$$

This formula represents the surface area of the solid formed by rotating the curve $x = g(y)$ about the y-axis.

Calculating the Surface Area of a Surface of Revolution 1

Let $f(x) = \sqrt{x}$ over the interval $[1, 4]$. Find the surface area of the surface generated by revolving the graph of $f(x)$ around the x-axis.

Solution: The graph of $f(x)$ and the surface of rotation are shown in the following figure.



We have $f(x) = \sqrt{x}$. Then, $f'(x) = \frac{1}{2\sqrt{x}}$ and $(f'(x))^2 = \frac{1}{4x}$. Then,

$$\text{Surface Area} = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\, dx = \int_1^4 2\pi\sqrt{x}\sqrt{1 + \frac{1}{4x}}\, dx$$

$$= \int_1^4 2\pi\sqrt{x + \frac{1}{4}}\, dx.$$

Solution

Let $u = x + \frac{1}{4}$. Then, $du = dx$. When $x = 1$, $u = \frac{5}{4}$, and when $x = 4$, $u = \frac{17}{4}$. This gives us

$$\int_0^1 2\pi\sqrt{x + \frac{1}{4}}\,dx = \int_{\frac{5}{4}}^{\frac{17}{4}} 2\pi\sqrt{u}\,du$$

$$= 2\pi\left[\frac{2}{3}u^{3/2}\right]\Bigg|_{\frac{5}{4}}^{\frac{17}{4}}$$

$$= \frac{\pi}{6}\left[17\sqrt{17} - 5\sqrt{5}\right].$$

Therefore, the surface area of the surface of revolution is $\frac{\pi}{6}\left[17\sqrt{17} - 5\sqrt{5}\right]$.

Calculating the Surface Area of a Surface of Revolution

Let $f(x) = \sqrt{1-x}$ over the interval $[0, \frac{1}{2}]$. Find the surface area of the surface generated by revolving the graph of $f(x)$ around the x-axis.

Solution: We have $f(x) = \sqrt{1-x}$. Then, $f'(x) = -\frac{1}{2\sqrt{1-x}}$ and $\left(f'(x)\right)^2 = \frac{1}{4(1-x)}$. Then,

$$\text{Surface Area} = \int_a^b 2\pi f(x)\sqrt{1+\left(f'(x)\right)^2}\,dx = \int_0^{\frac{1}{2}} 2\pi\sqrt{1-x}\sqrt{1+\frac{1}{4(1-x)}}\,dx$$

$$= \int_0^{\frac{1}{2}} \pi\sqrt{5-4x}\,dx$$

Let $u = 5-4x$, $du = -4dx$, for $x = 0, u = 5$ and for $x = 1/2$, $u = 3$

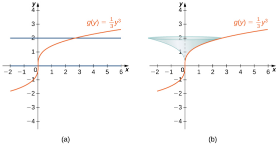
$$\text{Surface Area} = -\int_5^3 \frac{\pi}{4}\sqrt{u}\,du = -\frac{\pi}{4}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_5^3$$

$$= \frac{\pi}{6}\left(5\sqrt{5} - 3\sqrt{3}\right)$$

Calculating the Surface Area of a Surface of Revolution 2

Let $f(x) = y = \sqrt[3]{3x}$. Consider the portion of the curve where $0 \leq y \leq 2$. Find the surface area of the surface generated by revolving the graph of $f(x)$ around the y-axis.

Solution: Notice that we are revolving the curve around the y-axis, and the interval is in terms of y, so we want to rewrite the function as a function of y. We get $x = g(y) = \frac{1}{3}y^3$. The graph of $g(y)$ and the surface of rotation are shown in the following figure.



We have $g(y) = \frac{1}{3}y^3$, so $g'(y) = y^2$ and $(g'(y))^2 = y^4$. Then,

$$\text{Surface Area} = \int_c^d 2\pi g(y)\sqrt{1 + (g'(y))^2}\,dy = \int_0^2 2\pi\left(\frac{1}{3}y^3\right)\sqrt{1 + y^4}\,dy$$

$$= \frac{2\pi}{3}\int_0^2 y^3\sqrt{1 + y^4}\,dy.$$

Solution part 2

Let $u = y^4 + 1$. Then $du = 4y^3 dy$. When $y = 0$, $u = 1$, and when $y = 2$, $u = 17$. Then

$$\begin{aligned}\frac{2\pi}{3}\int_0^2 y^3\sqrt{1+y^4}\,dy &= \frac{2\pi}{3}\int_1^{17}\frac{1}{4}\sqrt{u}\,du \\ &= \frac{\pi}{6}\left[\frac{2}{3}u^{3/2}\right]\Bigg|_1^{17} \\ &= \frac{\pi}{9}\left[(17)^{3/2}-1\right].\end{aligned}$$

Therefore, the surface area of the surface of revolution is $\frac{\pi}{9}\left[(17)^{3/2}-1\right]$.

Calculating the Surface Area of a Surface of Revolution

Let $g(y) = \sqrt{9 - y^2}$ over the interval $y \in [0, 2]$. Find the surface area of the surface generated by revolving the graph of $g(y)$ around the y-axis.

Solution: Notice that we are revolving the curve around the y-axis. We are given the function $g(y) = \sqrt{9 - y^2}$, and we want to find the surface area generated by revolving this curve around the y-axis.

The surface area of a surface of revolution can be found using the formula:

$$\text{Surface Area} = \int_c^d 2\pi g(y)\sqrt{1 + (g'(y))^2}\, dy$$

We have $g(y) = \sqrt{9 - y^2}$, so $g'(y) = \frac{-y}{\sqrt{9-y^2}}$.

$$\text{Surface Area} = \int_0^2 2\pi\sqrt{9 - y^2}\sqrt{1 + \frac{y^2}{9 - y^2}}\, dy = \int_0^2 2\pi\sqrt{9 - y^2}\sqrt{\frac{9}{9 - y^2}}\, dy$$

$$= \int_0^2 2\pi \cdot 3\, dy = 6\pi y\Big|_0^2 = 12\pi.$$

Therefore, the surface area of the surface of revolution is 12

Key Concepts

- The arc length of a curve can be calculated using a definite integral.
- The arc length is first approximated using line segments, which generates a Riemann sum. Taking a limit then gives us the definite integral formula. The same process can be applied to functions of y.
- The concepts used to calculate the arc length can be generalized to find the surface area of a surface of revolution.
- The integrals generated by both the arc length and surface area formulas are often difficult to evaluate.

Key Equations

Arc Length of a Function of x:

$$\text{Arc Length} = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

Arc Length of a Function of y:

$$\text{Arc Length} = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy$$

Surface Area of a Function of x revolved about the x-axis:

$$\text{Surface Area} = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\, dx$$

Surface Area of a Function of x revolved about the y-axis:

$$\text{Surface Area} = \int_a^b 2\pi x\sqrt{1 + [f'(x)]^2}\, dx$$

Surface Area of a Function of y revolved about the y-axis:

$$\text{Surface Area} = \int_c^d 2\pi g(y)\sqrt{1 + [g'(y)]^2}\, dy$$

Surface Area of a Function of y revolved about the x-axis:

$$\text{Surface Area} = \int_c^d 2\pi y\sqrt{1 + [g'(y)]^2}\, dy$$