1.1 Approximating Areas

Clotilde Djuikem

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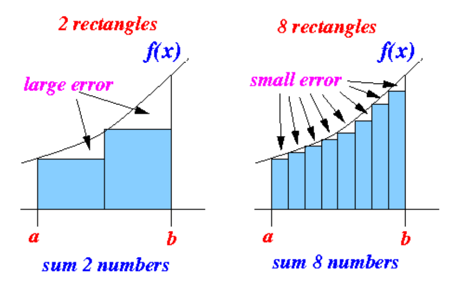
Outline

Motivation

Before

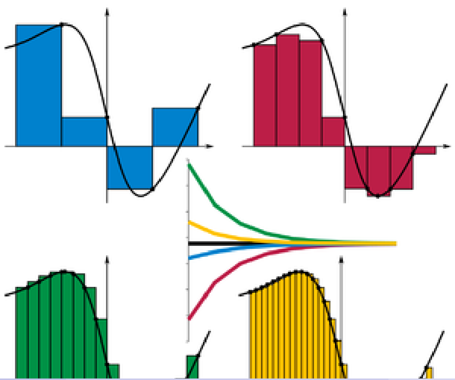
Imagine a bumpy field at a fair. We want to know how much space is there! Long ago, Archimedes used shapes to estimate areas. We do the same with rectangles. More rectangles mean a better guess.



Motivation

Today

Why do we do this? Think of planning a music festival. Calculating areas helps us organize spaces better. It is like having a secret tool for cool designs! We are learning these tricks to solve real-world puzzles someday. Is not that cool?



Learning Objectives

Objective 1

Use the sigma (summation) notation to calculate sums and powers of integers.

Objective 2

Use the sum of rectangular areas to approximate the area under a curve.

Objective 3

Use Riemann sums to approximate the area.

Sigma (Summation) Notation

In calculus, we use **sigma (Σ)** notation to make adding up lots of numbers easier.

Notation

For example, instead of writing $1 + 2 + 3 + \ldots + 19 + 20$,

we simply write $\sum_{i=1}^{20} i$.

Sigma notation looks like $\sum_{i=m}^{n} a_i$, where a_i are the terms to be added, i is the index of summation, and $m \leq n$ are the limits.

Let's try a couple of examples using sigma notation.

Example for Sigma

Using Sigma Notation

1. Write in sigma notation and evaluate the sum of terms 3^i for $i = 1, 2, 3, 4, 5$.
2. Write the sum in sigma notation: $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25}$.
3. Write in sigma notation and evaluate the sum of terms 2^i for i=3,4,5,6.

Example for Sigma

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3. Write in sigma notation and evaluate the sum of terms 2^i for i=3,4,5,6.

Solution

1. We have $\sum_{i=1}^{5} 3^i = 3 + 3^2 + 3^3 + 3^4 + 3^5 = 363$.
2. Using sigma notation, this sum can be written as $\sum_{i=1}^{5} \frac{1}{i^2}$.

Properties of Sigma Notation

Notation

Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ represent two sequences of terms and let c be a constant. The following properties hold for all positive integers n and for integers k, with $1 \leq k < n$.

$$\textbf{1.} \sum_{i=1}^{n} c = nc, \qquad \textbf{2.} \sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$$

$$\textbf{3.} \sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i, \qquad \textbf{4.} \sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$$

$$\textbf{5.} \sum_{i=1}^{n} a_i = \sum_{i=1}^{k} a_i + \sum_{i=k+1}^{n} a_i$$

Sums of Powers of Integers: To keep in mind

The sum of the first n integers is given by

$$\sum_{i=1}^{n} i = 1 + 2 + \ldots + n = \frac{n\,(n+1)}{2}$$

The sum of the squares of the first n integers is given by

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \ldots + n^2 = \frac{n\,(n+1)\,(2n+1)}{6}$$

The sum of the cubes of the first n integers is given by

$$\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4} = \left(\frac{n(n+1)}{2}\right)^2$$

Evaluation Using Sigma Notation

Write the following sums using sigma notation and then evaluate them.

1. The sum of the terms $(i-3)^2$ for $i = 1, 2, \ldots, 200$.
2. The sum of the terms $\left(i^3 - i^2\right)$ for $i = 1, 2, 3, 4, 5, 6$.

Solution 1

We expand $(i-3)^2$, and then use properties of sigma notation along with the summation formulas to obtain

$$\begin{aligned}\sum_{i=1}^{200}(i-3)^2 &= \sum_{i=1}^{200}\left(i^2-6i+9\right)\\ &= \sum_{i=1}^{200} i^2 - \sum_{i=1}^{200} 6i + \sum_{i=1}^{200} 9 \quad \text{(properties 3 and 4)}\\ &= \sum_{i=1}^{200} i^2 - 6\sum_{i=1}^{200} i + \sum_{i=1}^{200} 9 \quad \text{(property 2)}\\ &= \frac{200\,(200+1)\,(400+1)}{6} - 6\left[\frac{200\,(200+1)}{2}\right] + 9\,(200)\\ &= 2,686,700 - 120,600 + 1800\\ &= 2,567,900\end{aligned}$$

Solution 2

We use sigma notation property 4 and the formulas for the sum of squared terms and the sum of cubed terms to obtain

$$\begin{aligned}\sum_{i=1}^{6}\left(i^3-i^2\right) &= \sum_{i=1}^{6} i^3 - \sum_{i=1}^{6} i^2 \\ &= \frac{6^2(6+1)^2}{4} - \frac{6\,(6+1)\,(2\,(6)+1)}{6} \\ &= \frac{1764}{4} - \frac{546}{6} \\ &= 350\end{aligned}$$

Problem

Find the sum of the values of $(4 + 3i)$ for $i = 1, 2, \ldots, 100$.

Answer: 15,550

Hint: Use the properties of sigma notation to solve the problem.

Finding the Sum of the Function Values

Find the sum of the values of $f(x) = x^3$ over the integers $1, 2, 3, \ldots, 10$.

Finding the Sum of the Function Values

Find the sum of the values of $f(x) = x^3$ over the integers $1, 2, 3, \ldots, 10$.

Solution:

$$\sum_{i=1}^{10} i^3 = \frac{(10)^2(10+1)^2}{4} = \frac{100 \times 121}{4} = 3025.$$

Finding the Sum of a Linear Function

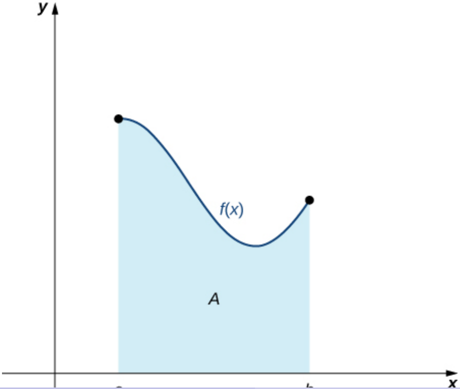
Let $f(x) = 2x + 1$. Evaluate the sum $\sum_{k=1}^{20} f(k)$.

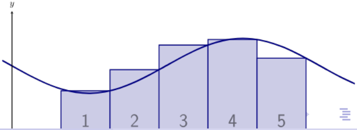
Answer: 440

Hint: Use the rules of sums and formulas for the sum of integers.

Problem

Now that we have the necessary notation, we return to the problem at hand: approximating the area under a curve. Let $f(x)$ be a continuous, nonnegative function defined on the closed interval $[a, b]$. We want to approximate the area A of the region under the curve $y = f(x)$, above the x-axis, and between the lines x=a and x=b, as shown on the figure below.





Idea

To approximate the area under the curve, we use a geometric approach. We divide the region into many small shapes, approximate each of them with a rectangle that has a known area formula, and then sum the areas of rectangles to obtain a reasonable estimate of the area of the region. We begin by dividing the interval $[a, b]$ into subintervals.

Definition

Consider an interval $[a, b]$. A set of points $P = \{x_i\}_{i=1}^{n}$ with $a = x_0 < x_1 < x_2 < \ldots < x_n = b$, which divides the interval $[a, b]$ into subintervals $[x_0, x_1]$, $[x_1, x_2]$, ..., $[x_{n-1}, x_n]$ is called a partition of $[a, b]$. If all the subintervals have the same width, the set of points forms a regular partition of the interval $[a, b]$.

For the regular partition, the width of each subinterval is denoted by Δx, so that

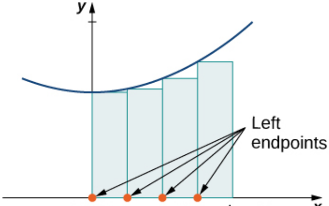
subinterval

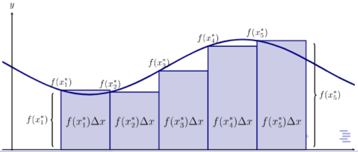
The subinterval $\Delta x = \frac{b-a}{n}$ and then $x_i = x_0 + i\Delta x$ for $i = 1, 2, 3, \ldots, n$

Left-Endpoint Approximation

On each subinterval $[x_{i-1}, x_i]$ $(i = 1, 2, 3, \ldots, n)$, construct a rectangle with a width of Δx and a height of $f(x_{i-1})$, the function value at the left endpoint of the subinterval. This ensures that the left upper corner of the rectangle belongs to the curve $y = f(x)$ (see Figure 2 below). This rectangle approximates the region below the graph of f over the subinterval $[x_{i-1}, x_i]$, and its area is $f(x_{i-1})\Delta x$.

$$A \approx L_n = f(x_0)\Delta x + f(x_1)\Delta x + \ldots + f(x_{n-1})\Delta x = \sum_{i=1}^{n} f(x_{i-1})\Delta x$$

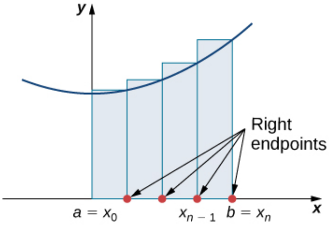


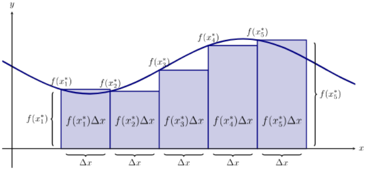


Right-Endpoint Approximation

Construct a rectangle on each subinterval $[x_{i-1}, x_i]$ $(i = 1, 2, 3, \ldots, n)$ with the height of $f(x_i)$, the function value at the right endpoint of the subinterval. This ensures that the right upper corner of the rectangle belongs to the curve $y = f(x)$ (see Figure 3 below).

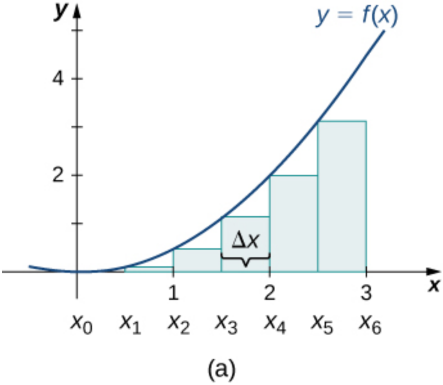
$$A \approx R_n = f(x_1)\Delta x + f(x_2)\Delta x + \ldots + f(x_n)\Delta x = \sum_{i=1}^{n} f(x_i)\Delta x.$$





Frame Title

In this Figure, the area of the region below the graph of the function $f(x) = \frac{x^2}{2}$ over the interval $[0, 3]$ is approximated using left- and right-endpoint approximations with six rectangles.



Left-Endpoint Approximation

In this case, $\Delta x = \dfrac{3-0}{6} = 0.5$, and the subintervals are $[0, 0.5]$, $[0.5, 1]$, $[1, 1.5]$, $[1.5, 2]$, $[2, 2.5]$, $[2.5, 3]$, that is, $x_0 = 0$, $x_1 = 0.5$, $x_2 = 1$, $x_3 = 1.5$, $x_4 = 2$, $x_5 = 2.5$, and $x_6 = 3$. Using the left-approximation formula for L_n, we obtain

$$\begin{aligned} A &\approx L_6 = \sum_{i=1}^{6} f(x_{i-1})\Delta x \\ &= f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x + f(x_5)\Delta x \\ &= f(0)\cdot 0.5 + f(0.5)\cdot 0.5 + f(1)\cdot 0.5 + f(1.5)\cdot 0.5 + f(2)\cdot 0.5 + f(2.5) \\ &= 0\cdot 0.5 + 0.125\cdot 0.5 + 0.5\cdot 0.5 + 1.125\cdot 0.5 + 2\cdot 0.5 + 3.125\cdot 0.5 \\ &= 0 + 0.0625 + 0.25 + 0.5625 + 1 + 1.5625 \\ &= 3.4375. \end{aligned}$$

Right-Endpoint Approximation

Using the right-approximation formula for R_n, we obtain

$$\begin{aligned} A &\approx R_6 = \sum_{i=1}^{6} f(x_i)\Delta x \\ &= f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x + f(x_5)\Delta x + f(x_6)\Delta x \\ &= f(0.5)\cdot 0.5 + f(1)\cdot 0.5 + f(1.5)\cdot 0.5 + f(2)\cdot 0.5 + f(2.5)\cdot 0.5 + f(3) \\ &= 0.125\cdot 0.5 + 0.5\cdot 0.5 + 1.125\cdot 0.5 + 2\cdot 0.5 + 3.125\cdot 0.5 + 4.5\cdot 0.5 \\ &= 0.0625 + 0.25 + 0.5625 + 1 + 1.5625 + 2.25 \\ &= 5.6875. \end{aligned}$$

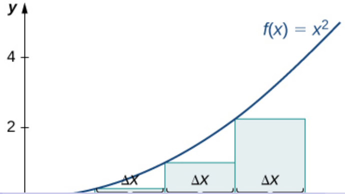
Approximating the Area Under a Curve

Use both left- and right-endpoint approximations to approximate the area under the graph of $f(x) = x^2$ over the interval $[0, 2]$ using $n = 4$.

Solution - Left-Endpoint Approximation

First, divide the interval $[0, 2]$ into n equal subintervals. Using $n = 4$, $\Delta x = \frac{(2-0)}{4} = 0.5$. This is the width of each rectangle. The intervals $[0, 0.5]$, $[0.5, 1]$, $[1, 1.5]$, $[1.5, 2]$ are shown in Figure 5. Using the left-endpoint approximation, the heights are $f(0) = 0$, $f(0.5) = 0.25$, $f(1) = 1$, $f(1.5) = 2.25$. Then,

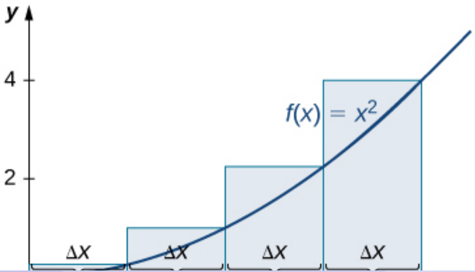
$$\begin{aligned} L_4 &= f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x \\ &= 0 \cdot 0.5 + 0.25 \cdot 0.5 + 1 \cdot 0.5 + 2.25 \cdot 0.5 \\ &= 1.75. \end{aligned}$$



Solution: Right-Endpoint Approximation

The right-endpoint approximation is shown in Figure 6. The intervals are the same, $\Delta x = 0.5$, but now we use the right endpoints to calculate the heights of the rectangles. We have

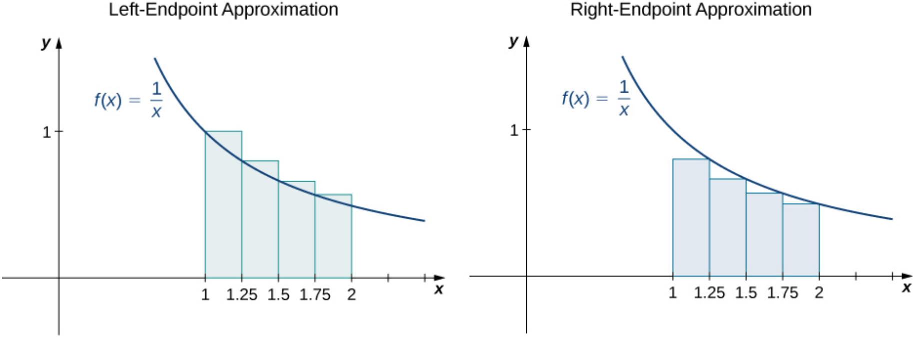
$$\begin{aligned} R_4 &= f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x \\ &= 0.25 \cdot 0.5 + 1 \cdot 0.5 + 2.25 \cdot 0.5 + 4 \cdot 0.5 \\ &= 3.75. \end{aligned}$$



Sketch Left- and Right-Endpoint Approximations

Sketch left- and right-endpoint approximations for $f(x) = \frac{1}{x}$ on $[1, 2]$ using $n = 4$. Approximate the area using both methods.

Solution The left-endpoint approximation is 0.7595. The right-endpoint approximation is 0.6345. See the figure below.



(a) (b)

Generalizing Approximations

So far, to approximate the area under a curve, we have been using rectangles with the heights determined by evaluating the function at either the left or the right endpoint of the subinterval $[x_{i-1}, x_i]$. However, we could evaluate the function at any point x_i^* in $[x_{i-1}, x_i]$, and use $f(x_i^*)$ as the height of the approximating rectangle. This would result in an estimate $A \approx \sum_{i=1}^{n} f(x_i^*)\Delta x$.

Riemann Sum

Let the function $f(x)$ be defined on a closed interval $[a, b]$ and let P be a regular partition of $[a, b]$ with the subinterval width Δx. For each $1 \leq i \leq n$, let x_i^* be an arbitrary point in $[x_{i-1}, x_i]$. The numbers $x_1^*, x_2^*, \ldots, x_n^*$ are called the sample points. Then the Riemann sum for $f(x)$ that corresponds to the partition P and the set of sample points $\{x_i^*\}_{i=1}^n$ is defined as

$$\sum_{i=1}^{n} f(x_i^*)\Delta x.$$

Definition: Area Under the Curve

Let $f(x)$ be a continuous, nonnegative function on an interval $[a, b]$, and let $\sum_{i=1}^{n} f(x_i^*)\Delta x$ be a Riemann sum for $f(x)$. Then, the area under the curve $y = f(x)$ over $[a, b]$ is given by

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x.$$

Finding Lower Sums

Problem: Find the lower sum for $f(x) = 10 - x^2$ over $[1, 2]$ with $n = 4$

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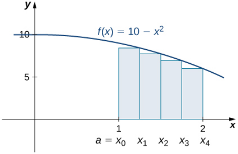


Solution:

$$\Delta x = \frac{2-1}{4} = \frac{1}{4},$$

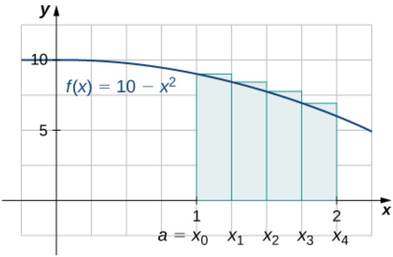
$$R_4 = \sum_{k=1}^{4} (10 - x_i^2) \cdot 0.25$$

$$= 0.25\,[8.4375 + 7.75 + 6.9375 + 6]$$

$$= 7.28.$$

Hence, the lower sum is 7.28.

Finding Upper Sums

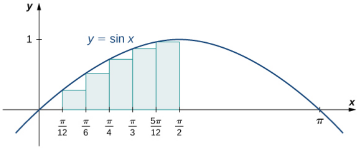


Hence, the upper sum is 8.0313.

Hint: $f(x)$ is decreasing on $[1, 2]$, so the maximum function values occur at the left endpoints of the subintervals.

Finding Lower Sums

Problem: Find the lower sum for $f(x) = \sin(x)$ over $[0, \pi/2]$ with $n = 6$ subintervals.



Solution:

$$\Delta x = \frac{\pi/2 - 0}{6} = \frac{\pi}{12},$$

$$L_6 = \frac{\pi}{12}\left[0 + \sin\left(\frac{\pi}{12}\right) + \frac{1}{2} + \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} + \sin\left(\frac{5\pi}{12}\right)\right]$$

$$= \frac{\pi(1 + \sqrt{2} + \sqrt{3} + \sqrt{6})}{24}.$$

Finding Upper Sums

Problem: Find the upper sum for $f(x) = \sin(x)$ over $[0, \pi/2]$ with $n = 6$ subintervals.
Solution:

$$\Delta x = \frac{\pi/2 - 0}{6} = \frac{\pi}{12},$$
$$R_6 = \frac{\pi(3 + \sqrt{2} + \sqrt{3} + \sqrt{6})}{24}.$$

Hint: Compare the expressions for the upper and lower sums.

1.2 The Definite Integral

Clotilde Djuikem

January 23, 2024

Outline

Learning Objectives

1. State the definition of the definite integral.
2. Explain the terms integrand, limits of integration, and variable of integration.
3. Explain when a function is integrable.
4. Describe the relationship between the definite integral and net area.
5. Use geometry and the properties of definite integrals to evaluate them.
6. Calculate the average value of a function.

Reminder

In the preceding section, we defined the area under a curve in terms of Riemann sums:

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x.$$

However, this definition came with restrictions. We required $f(x)$ to be continuous and nonnegative.

Extension of the concept

Real-world problems often do not adhere to these restrictions. In this section, we explore extending the concept of the area under the curve to a wider range of functions using the definite integral.

Definition

If $f(x)$ is a function defined on an interval $[a, b]$, the definite integral of f from a to b is given by

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x,$$

provided the limit exists.

If this limit exists, the function $f(x)$ is said to be integrable on $[a, b]$, or is an integrable function.

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Notation

The function $f(x)$ is the integrand, and the dx called the variable of integration. Note that, like the index in a sum, the variable of integration is a dummy variable, and has no impact on the computation of the integral.

Theorem

We could use any variable we like as the variable of integration:

$$\int_a^b f(x)\ dx = \int_a^b f(t)\ dt = \int_a^b f(u)\, du$$

Theorem

If $f(x)$ is continuous on $[a, b]$, then f is integrable on $[a, b]$.

Remark

Functions that are not continuous on $[a, b]$ may still be integrable, depending on the nature of the discontinuities. For example, functions with a finite number of jump discontinuities on a closed interval are integrable.

Evaluation of Definite Integral

Problem: Evaluate $\int_0^2 x^2\,dx$ using the definition of the definite integral. Utilize a right-endpoint approximation to generate the Riemann sum.

Evaluation of Definite Integral

Problem: Evaluate $\int_0^2 x^2\,dx$ using the definition of the definite integral. Utilize a right-endpoint approximation to generate the Riemann sum.

Solution:

$$\Delta x = \frac{b-a}{n} = \frac{2}{n}, \quad \text{where } a = 0, b = 2$$

$$x_i = \frac{2i}{n}, \quad \text{for } i = 1, 2, \ldots, n;\ f(x_i) = \left(\frac{2i}{n}\right)^2 = \frac{4i^2}{n^2}$$

$$\sum_{i=1}^{n} f(x_i)\Delta x = \frac{8}{n^3}\sum_{i=1}^{n} i^2 = \frac{8}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right] = \frac{8}{n^3}\left[\frac{2n^3+3n^2+n}{6}\right]$$

To calculate the definite integral, take the limit as $n \to \infty$:

$$\int_0^2 x^2\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty}\left(\frac{8}{3}+\frac{4}{n}+\frac{1}{6n^2}\right) = \frac{8}{3}$$

Evaluation of Definite Integral

Problem: Evaluate $\int_0^3 (2x - 1)\, dx$ using the definition of the definite integral. Utilize a right-endpoint approximation to generate the Riemann sum.

Evaluation of Definite Integral

Problem: Evaluate $\int_0^3 (2x-1)\,dx$ using the definition of the definite integral. Utilize a right-endpoint approximation to generate the Riemann sum. **Solution:**

$$\Delta x = \frac{b-a}{n} = \frac{3}{n}, \quad \text{where } a=0, b=3;\ x_i = \frac{3i}{n},\ \text{for } i = 1, 2, \ldots, n$$

$$f(x_i) = 2x_i - 1 = 2\left(\frac{3i}{n}\right) - 1 = \frac{6i}{n} - 1$$

$$\sum_{i=1}^{n} f(x_i)\Delta x = \frac{18}{n^2}\sum_{i=1}^{n} i - \frac{3}{n}\sum_{i=1}^{n} 1 = \frac{18}{n^2}\left[\frac{n(n+1)}{2}\right] - \frac{3}{n}\sum_{i=1}^{n} 1$$

$$= \frac{18}{n^2}\left[\frac{n^2+n}{2}\right] - \frac{3}{n}(n) = \frac{18}{2} + \frac{18}{2n} - 3$$

$$\int_0^3 (2x-1)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty}\left(\frac{18}{2} + \frac{18}{2n} - 3\right) = 6$$

Evaluation of Definite Integral

Problem: Set up and expression for $\int_0^3 (e^x - 1)\, dx$. Use the right endpoint and do not evaluate.

Evaluation of Definite Integral

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$$\Delta x = \frac{b-a}{n} = \frac{3}{n}, \quad \text{where } a = 0, b = 3$$

$$x_i = \frac{3i}{n}, \quad \text{for } i = 1, 2, \ldots, n$$

$$f(x_i) = e^{x_i} - 1 = e^{\frac{3i}{n}} - 1$$

$$\sum_{i=1}^{n} f(x_i)\Delta x = \frac{3}{n}\sum_{i=1}^{n}\left(e^{\frac{3i}{n}} - 1\right)$$

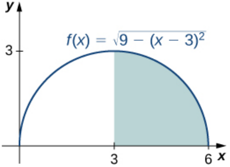
To calculate the definite integral, take the limit as $n \to \infty$:

$$\int_0^3 (e^x - 1)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty}\left[\frac{3}{n}\sum_{i=1}^{n}\left(e^{\frac{3i}{n}} - 1\right)\right]$$

Using Geometric Formulas to Calculate Definite Integrals

Problem: Use the formula for the area of a circle to evaluate $\int_3^6 \sqrt{9-(x-3)^2}\,dx$.

Solution: The function describes a semicircle with radius 3. To find



we want to find the area under the curve over the interval $[3, 6]$. The formula for the area of a circle is $A = \pi r^2$. The area of a semicircle is just one-half the area of a circle, or $A = \left(\frac{1}{2}\right)\pi r^2$. The shaded area in the above Figure covers one-half of the semicircle, or $A = \left(\frac{1}{4}\right)\pi r^2$.

$$\int_3^6 \sqrt{9-(x-3)^2}\,dx = \frac{1}{4}\pi(3)^2 = \frac{9}{4}\pi$$

Using Geometric Formulas to Calculate Definite Integrals

Problem: Use the formula for the area of a trapezoid to evaluate $\int_2^4 (2x+3)\,dx$.

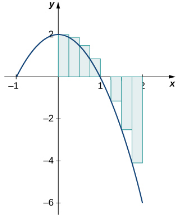
Solution: The given function represents the height of a trapezoid. To find the area under the curve over the interval $[2, 4]$, we can use the formula for the area of a trapezoid:

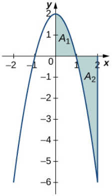
$$A = \frac{1}{2}h(b_1 + b_2)$$

where h is the height and b_1, b_2 are the bases.
Substituting the values:

$$A = \frac{1}{2}(3)(2 + (2 \cdot 4 + 3)) = 18 \text{ square units}$$

Net Area





$$\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x = (\text{Area of rectangles above the } x\text{-axis}) - (\text{Area of rectangles below the } x\text{-axis})$$

Net signed and total area

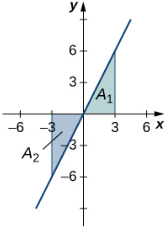
In the case where the function in integrable on $[a, b]$

$$\int_a^b f(x)dx = A_1 - A_2 \text{ and } \int_a^b |f(x)|\, dx = A_1 + A_2.$$

Finding the Net Signed Area

Problem: $f(x) = 2x$ and the x-axis over the interval $[-3, 3]$.

Solution: The function produces a straight line that forms two triangles: one from $x = -3$ to $x = 0$ and the other from $x = 0$ to $x = 3$,



Using the geometric formula for the area of a triangle, $A = \frac{1}{2}bh$, the area of triangle A_1, above the axis, is $A_1 = \frac{1}{2}(3)(6) = 9$,. The area of triangle A_2, below the axis, is $A_2 = \frac{1}{2}(3)(6) = 9$, Thus, the net area is

$$\int_{-3}^{3} 2x\,dx = A_1 - A_2 = 9 - 9 = 0.$$

Properties of the Definite Integral

Suppose that the functions f and g are integrable over all given intervals.

$$\int_a^a f(x)\,dx = 0; \quad \int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

$$\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

.

$$\int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$$

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Using the Properties of the Definite Integral

Problem: Express $\int_{-2}^{1}(-3x^3+2x+2)\,dx$ as the sum of three definite integrals using the properties of the definite integral.
Solution: Using integral notation, we have

$$\int_{-2}^{1}(-3x^3+2x+2)\,dx.$$

We apply properties 3 and 5 to get

$$\begin{aligned}\int_{-2}^{1}(-3x^3+2x+2)\,dx &= \int_{-2}^{1}-3x^3\,dx+\int_{-2}^{1}2x\,dx+\int_{-2}^{1}2\,dx\\ &= -3\int_{-2}^{1}x^3\,dx+2\int_{-2}^{1}x\,dx+\int_{-2}^{1}2\,dx.\end{aligned}$$

Using the Properties of the Definite Integral

Problem: Express $\int_1^3 (6x^3 - 4x^2 + 2x - 3)\, dx$ as the sum of four definite integrals using the properties of the definite integral.

Using the Properties of the Definite Integral

Problem: Express $\int_1^3 (6x^3 - 4x^2 + 2x - 3)\,dx$ as the sum of four definite integrals using the properties of the definite integral. **Solution:** Using integral notation, we have

$$\int_1^3 (6x^3 - 4x^2 + 2x - 3)\,dx.$$

We apply properties to express it as the sum of four definite integrals:

$$\int_1^3 (6x^3 - 4x^2 + 2x - 3)\,dx \quad = 6\int_1^3 x^3\,dx - 4\int_1^3 x^2\,dx + 2\int_1^3 x\,dx - \int_1^3 3\,dx$$

Using the Properties of the Definite Integral

Problem: If it is known that $\int_0^8 f(x)\,dx = 10$ and $\int_0^5 f(x)\,dx = 5$, find the value of $\int_5^8 f(x)\,dx$.
Solution: By property 6,

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Thus,

$$\begin{aligned}\int_0^8 f(x)\,dx &= \int_0^5 f(x)\,dx + \int_5^8 f(x)\,dx\\ 10 &= 5 + \int_5^8 f(x)\,dx\\ 5 &= \int_5^8 f(x)\,dx.\end{aligned}$$

Using the Properties of the Definite Integral

Problem: If it is known that $\int_1^5 f(x)\,dx = -3$ and $\int_2^5 f(x)\,dx = 4$, find the value of $\int_1^2 f(x)\,dx$.
Solution: By property 6,

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Thus,

$$\begin{aligned} \int_1^5 f(x)\,dx &= \int_1^2 f(x)\,dx + \int_2^5 f(x)\,dx \\ -3 &= \int_1^2 f(x)\,dx + 4 \\ -7 &= \int_1^2 f(x)\,dx. \end{aligned}$$

Comparison Theorem

Suppose that the functions $f(x)$ and $g(x)$ are integrable over the interval $[a, b]$.

If $f(x) \geq 0$ for $a \leq x \leq b$, then

$$\int_a^b f(x)\,dx \geq 0.$$

If $f(x) \geq g(x)$ for $a \leq x \leq b$, then

$$\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx.$$

If m and M are constants such that $m \leq f(x) \leq M$ for $a \leq x \leq b$, then

$$\begin{aligned} m(b-a) &\leq \int_a^b f(x)\,dx \\ &\leq M(b-a). \end{aligned}$$

Comparing Integrals over a Given Interval

Problem: Compare the integrals of the functions $f(x) = \sqrt{1+x^2}$ and $g(x) = \sqrt{1+x}$ over the interval $[0, 1]$.
Solution: Comparing functions $f(x)$ and $g(x)$ when $x \in [0, 1]$. Since $1 + x^2 \geq 0$ and $1 + x \geq 0$ for $x \in [0, 1]$, comparing $\sqrt{1+x^2}$ and $\sqrt{1+x}$ is equivalent to comparing the expressions $(1 + x^2)$ and $(1 + x)$ under the roots on $[0, 1]$. We consider :

$$(1 + x^2) - (1 + x) = 1 + x^2 - 1 - x = x^2 - x = x(x - 1).$$

Since $x \geq 0$ and $x - 1 \leq 0$ on $[0, 1]$, we have that $x(x - 1) \leq 0$ on $[0, 1]$. It follows that $1 + x^2 \leq 1 + x$ on $[0, 1]$, and hence

$$f(x) = \sqrt{1+x^2} \leq \sqrt{1+x} = g(x), \quad x \in [0, 1].$$

Since both functions $f(x)$ and $g(x)$ are continuous on $[0, 1]$,

$$\int_0^1 f(x)\,dx \leq \int_0^1 g(x)\,dx.$$

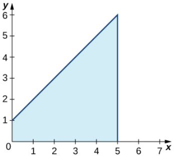
Definition

Let $f(x)$ be continuous over the interval $[a, b]$. Then, the average value of the function $f(x)$ (denoted by f_{ave}) on $[a, b]$ is given by

$$f_{\text{ave}} = \frac{1}{b-a} \int_a^b f(x)\, dx.$$

Finding the Average Value of a Linear Function

Problem: Find the average value of $f(x) = x + 1$ over the interval $[0, 5]$.
Solution: First, graph the function on the stated interval, as shown below.



The region is a trapezoid lying on its side, so we can use the area formula for a trapezoid $A = \frac{1}{2}h(a+b)$, where h represents height, and a and b represent the two parallel sides. Then,

$$\int_0^5 (x+1)\,dx \quad = \tfrac{1}{2}h(a+b) = \tfrac{1}{2}\cdot 5 \cdot (1+6) = \tfrac{35}{2}.$$

Thus, the average value of the function is

$$\frac{1}{5}\int_0^5 (x+1)\,dx = \frac{1}{5}\cdot\frac{35}{2} = \frac{7}{2}.$$

Finding the Average Value of a Linear Function

Problem: Find the average value of $f(x) = 6 - 2x$ over the interval $[0, 3]$.
Solution: Use the average value formula and geometry to evaluate the integral. First, note that the function is a linear function, representing a downward-sloping line.
Apply the average value formula:

$$\text{Average Value} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

$$\begin{aligned}\int_0^3 (6-2x)\,dx &= \tfrac{1}{3-0}\int_0^3 (6-2x)\,dx \\ &= \tfrac{1}{3}\left[6x - x^2\right]_0^3 \\ &= \tfrac{1}{3}[(18-9)-(0-0)] \\ &= \tfrac{9}{3} = 3.\end{aligned}$$

1.3 The Fundamental Theorem of Calculus

Clotilde Djuikem

January 30, 2024

Outline

Learning Objectives

1. Describe the meaning of the Mean Value Theorem for Integrals.
2. State the meaning of the Fundamental Theorem of Calculus, Part 1.
3. Use the Fundamental Theorem of Calculus, Part 1, to evaluate derivatives of integrals.
4. Review the notions of an Antiderivative and an Indefinite Integral, the Table of Antiderivatives, and the Properties of Indefinite Integrals.
5. State the meaning of the Fundamental Theorem of Calculus, Part 2.
6. Use the Fundamental Theorem of Calculus, Part 2, to evaluate definite integrals.
7. Explain the relationship between differentiation and integration.

Mean Value Theorem for Integrals

If $f(x)$ is continuous over an interval $[a, b]$, then there is at least one point $c \in [a, b]$ such that

$$f(c) = \frac{1}{b-a} \int_a^b f(x)\,dx.$$

This formula can also be stated as

$$\int_a^b f(x)\,dx = f(c) \cdot (b-a).$$

Proof

Since $f(x)$ is continuous on $[a, b]$, by the extreme value theorem, it assumes min and max values m and M, on $[a, b]$. $\forall x$ in $[a, b]$, we have $m \leq f(x) \leq M$. Therefore, by the comparison theorem, we have

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a).$$

Dividing by $b-a$ gives us

$$m \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq M.$$

Since $\frac{1}{b-a}\int_a^b f(x)\,dx$ is a number between m and M, and since $f(x)$ is continuous and assumes the values m and M over $[a, b]$, by the Intermediate Value Theorem, there is a number c in $[a, b]$ such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx,$$

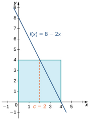
Finding the Average Value of a Function

Find the average value of the function $f(x) = 8 - 2x$ on $[0, 4]$ and find c such that $f(c)$ equals the average value of the function over $[0, 4]$.

Solution

The formula states the mean value of $f(x)$ is given by

$$\frac{1}{4-0}\int_0^4 (8-2x)\,dx.$$



The area of the triangle is $A = \frac{1}{2}(\text{base})(\text{height})$. We have $A = \frac{1}{2}(4)(8) = 16$.

The average value is found by multiplying the area by $\frac{1}{4-0}$. Thus, the average value of the function is $\frac{1}{4}(16) = 4$.

Set the average value equal to $f(c)$ and solve for c.

$$8 - 2c = 4,\ c = 2 \text{ Then At} c = 2, f(2) = 4.$$

Finding Average Value - Solution (Part 1)

Problem: Find the average value of the function $f(x) = \frac{x}{2}$ over the interval $[0, 6]$ and find c such that $f(c)$ equals the average value of the function over $[0, 6]$.

Solution: The formula for the mean value of $f(x)$ over the interval $[a, b]$ is given by

$$\text{Average value} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

For this problem, $a = 0$, $b = 6$, and $f(x) = \frac{x}{2}$. Therefore,

$$\text{Average value} = \frac{1}{6}\int_0^6 \frac{x}{2}\,dx.$$

Finding Average Value - Solution (Part 2)

Solving the integral,

$$\text{Average value} = \frac{1}{6}\left[\frac{x^2}{4}\right]_0^6 = \frac{1}{6}\left(\frac{36}{4} - \frac{0}{4}\right) = \frac{1}{6}\cdot 9 = 1.5.$$

To find c such that $f(c)$ equals the average value, we set up the equation $f(c) = 1.5$:

$$\frac{c}{2} = 1.5.$$

Solving for c,

$$c = 3.$$

Therefore, the average value is 1.5, and c is 3.

Fundamental Theorem of Calculus, Part 1

If $f(x)$ is continuous over an interval $[a, b]$, and the function $F(x)$ is defined by

$$F(x) = \int_a^x f(t)dt, \text{ then } F'(x) = f(x) \text{ over } [a, b].$$

Proof: Applying the definition of the derivative, we have

$$F'(x) = \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t)dt.$$

we see that $\frac{1}{h} \int_x^{x+h} f(t)dt$ is just the average value of the function $f(x)$ on $[x, x+h]$. Therefore, by the mean value theorem for integrals, there is some number c in $[x, x+h]$ such that

$$\frac{1}{h}\int_{x}^{x+h} f(x)\,dx = f(c).$$

Since c approaches x as h approaches zero, and $f(x)$ is continuous, we have

$$\lim_{h\to 0} f(c) = \lim_{c\to x} f(c) = f(x).$$

Putting all these pieces together, we have

$$F'(x) = \lim_{h\to 0}\frac{1}{h}\int_{x}^{x+h} f(x)\,dx = \lim_{h\to 0} f(c) = f(x),$$

and the proof is complete. □

Finding a Derivative with the Fundamental Theorem of Calculus

Problem: Find the derivative of $g(x) = \int_1^x \frac{1}{t^3+1}\,dt$.

Solution

According to the Fundamental Theorem of Calculus, the derivative is given by

$$g'(x) = \frac{1}{x^3+1}.$$

Using the Fundamental Theorem of Calculus, Part 1

Problem: Use the Fundamental Theorem of Calculus, Part 1, to find the derivative of $g(r) = \int_0^r \sqrt{x^2 + 4}\, dx$.

Answer

$$g'(r) = \sqrt{r^2 + 4}.$$

Using the Fundamental Theorem and the Chain Rule

Problem: Let $F(x) = \int_1^{\sqrt{x}} \sin(t)\,dt$. Find $F'(x)$.

Fundamental Theorem of Calculus and the Chain Rule:

Let $F(x) = \int_a^{u(x)} f(t)\,dt$ be a function defined by an integral, where $u(x)$ is differentiable. Then, $F'(x) = f(u(x)) \cdot u'(x)$.

Using the Fundamental Theorem and the Chain Rule

Problem: Let $F(x) = \int_1^{\sqrt{x}} \sin(t)\,dt$. Find $F'(x)$.

Fundamental Theorem of Calculus and the Chain Rule:

Let $F(x) = \int_a^{u(x)} f(t)\,dt$ be a function defined by an integral, where $u(x)$ is differentiable. Then, $F'(x) = f(u(x)) \cdot u'(x)$.

Solution

Letting $u(x) = \sqrt{x}$, we have $F(x) = \int_1^{u(x)} \sin(t)dt$. Thus, by the Fundamental Theorem of Calculus and the chain rule,

$$F'(x) = \sin(u(x))\frac{du}{dx} = \sin(u(x)) \cdot \left(\frac{1}{2}x^{-1/2}\right) = \frac{\sin\sqrt{x}}{2\sqrt{x}}.$$

Fundamental Theorem of Calculus and the Chain Rule

Problem: Let $F(x) = \int_1^{x^3} \cos(t)\,dt$. Find $F'(x)$.

Solution

Let $u(x) = x^3$. Then, $F(x) = \int_1^{u(x)} \cos(t)\,dt$. According to the Fundamental Theorem of Calculus and the Chain Rule,

$$F'(x) = \cos(u(x)) \cdot u'(x).$$

Now, compute the derivatives:

$$u'(x) = 3x^2 \quad \text{and} \quad \cos(u(x)) = \cos(x^3).$$

Therefore, $F'(x) = 3x^2 \cdot \cos(x^3)$.

Using the Fundamental Theorem of Calculus with Two Variable Limits

Problem: Let $F(x) = \int_x^{2x} t^3\,dt$. Find $F'(x)$.

Solution

Since both limits of integration are variable, we split it into two integrals:

$$F(x) = \int_x^0 t^3\,dt + \int_0^{2x} t^3\,dt = -\int_0^x t^3\,dt + \int_0^{2x} t^3\,dt.$$

Differentiating the first term:

$$\frac{d}{dx}\left[-\int_0^x t^3\,dt\right] = -x^3.$$

solution Part 2

Solution

Thus,

$$F'(x) = \frac{d}{dx}\left[-\int_0^x t^3\,dt\right] + \frac{d}{dx}\left[\int_0^{2x} t^3\,dt\right]$$
$$= -x^3 + 16x^3$$
$$= 15x^3.$$

Finding the Derivative

Problem: Let $F(x) = \int_x^{x^2} \cos(t)\,dt$. Find $F'(x)$.

Solution

We have $F(x) = \int_x^{x^2} \cos(t)\,dt$. To find $F'(x)$, we apply the Fundamental Theorem of Calculus.

$$F'(x) = \cos(x^2) \cdot (x^2)' - \cos(x) \cdot (x)' = 2x\cos(x^2) - \cos(x).$$

Therefore,

$$F'(x) = 2x\cos(x^2) - \cos(x).$$

Definition: Antiderivative

A function F is an antiderivative of the function f over an interval I if $F'(x) = f(x)$ for all x in I.

- Unlike the derivative, if an antiderivative of a given function exists, it is not unique.
- If F is an antiderivative of f over an interval I, then the set of all antiderivatives of f over I, also called the most general antiderivative of f over I, has the form $F(x) + C$, where $C \in \mathbb{R}$ is an arbitrary constant.

The indefinite integral $\int f(x)\,dx$ is the notation used for the most general antiderivative of the function f on its domain:

$$\int f(x)\,dx = F(x) + C,$$

where F is any particular antiderivative of f on its domain, and C is an arbitrary constant.

Integration and Differentiation Formulas part 1

Differentiation Formulas:

$$\frac{d}{dx}(k) = 0$$
$$\frac{d}{dx}(x^n) = nx^{n-1}$$
$$\frac{d}{dx}(\ln|x|) = \frac{1}{x}$$
$$\frac{d}{dx}(e^x) = e^x$$
$$\frac{d}{dx}(a^x) = a^x \ln(a) \quad \text{for } a > 0, a \neq 1$$
$$\frac{d}{dx}(\sin(x)) = \cos(x)$$
$$\frac{d}{dx}(\cos(x)) = -\sin(x)$$

Indefinite Integrals:

$$\int k\, dx = kx + C$$
$$\int x^n\, dx = \frac{x^{n+1}}{n+1} + C \quad \text{for } n \neq -1$$
$$\int \frac{1}{x}\, dx = \ln|x| + C$$
$$\int e^x\, dx = e^x + C$$
$$\int a^x\, dx = \frac{a^x}{\ln(a)} + C \quad \text{for } a > 0, a \neq 1$$
$$\int \cos(x)\, dx = \sin(x) + C$$
$$\int \sin(x)\, dx = -\cos(x) + C$$

Integration and Differentiation Formulas part 2

Differentiation Formulas:

$$\frac{d}{dx}(\tan(x)) = \sec^2(x)$$
$$\frac{d}{dx}(\csc(x)) = -\csc(x)\cot(x)$$
$$\frac{d}{dx}(\sec(x)) = \sec(x)\tan(x)$$
$$\frac{d}{dx}(\cot(x)) = -\csc^2(x)$$
$$\frac{d}{dx}(\sin^{-1}(x)) = \frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}(\tan^{-1}(x)) = \frac{1}{1+x^2}$$
$$\frac{d}{dx}(\sec^{-1}|x|) = \frac{1}{x\sqrt{x^2-1}}$$

Indefinite Integrals:

$$\int \sec^2(x)\,dx = \tan(x) + C$$
$$\int \csc x\cot(x)\,dx = -\csc(x) + C$$
$$\int \sec x\tan(x)\,dx = \sec(x) + C$$
$$\int \csc^2(x)\,dx = -\cot(x) + C$$
$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}(x) + C$$
$$\int \frac{1}{1+x^2}\,dx = \tan^{-1}(x) + C$$
$$\int \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1}|x| + C$$

Properties of Indefinite Integrals

Sum and Difference Rules:

$$\int (f(x) \pm g(x))\, dx = F(x) \pm G(x) + C$$

Constant Multiple Rule:

$$\int (k \cdot f(x))\, dx = k \cdot F(x) + C$$

Note

There are **NO** product and quotient rules for indefinite integrals.

Evaluation: $\int (5x^3 - 7x^2 + 3x + 4)\, dx$

$$\int (5x^3 - 7x^2 + 3x + 4)\, dx = \frac{5}{4}x^4 - \frac{7}{3}x^3 + \frac{3}{2}x^2 + 4x + C$$

Evaluation: $\int \frac{x^2+4\sqrt[3]{x}}{x}\,dx$

$$\int \frac{x^2+4\sqrt[3]{x}}{x}\,dx = \frac{1}{2}x^2 + 12x^{1/3} + C$$

Evaluation: $\int \frac{4}{1+x^2}\,dx$

$$\int \frac{4}{1+x^2}\,dx = 4\tan^{-1}(x) + C$$

Evaluation: $\int \tan(x)\cos(x)\,dx$

$$\int \tan(x)\cos(x)\,dx = -\cos(x) + C$$

Problem

Evaluate the following indefinite integral:

$$\int (4x^3 - 5x^2 + e^x - 7)\, dx$$

Solution

Using the properties of indefinite integrals together with an antiderivative of a power function and the exponential function, we obtain

$$\int (4x^3 - 5x^2 + e^x - 7)\,dx = x^4 - \frac{5}{3}x^3 + e^x - 7x + C.$$

The Fundamental Theorem of Calculus, Part 2

If f is continuous over the interval $[a, b]$ and $F(x)$ is any antiderivative of $f(x)$ on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Proof

Let $P = \{x_i\}$, $i = 0, 1, \ldots, n$ be a regular partition of $[a, b]$. Then, we can write

$$F(b) - F(a) = F(x_n) - F(x_0) = [F(x_n) - F(x_{n-1})] + [F(x_{n-1}) - F(x_{n-2})] + \ldots + [F(x_1) - F(x_0)].$$

Proof of the Fundamental Theorem of Calculus, Part 2

Now, we know F is an antiderivative of f over $[a, b]$, and so F is an antiderivative of f over each $[x_{i-1}, x_i]$. Applying the Mean Value Theorem for integrals to f over $[x_{i-1}, x_i]$ for $i = 0, 1, \ldots, n$, we can find c_i in $[x_{i-1}, x_i]$ such that

$$F(x_i) - F(x_{i-1}) = F'(c_i)(x_i - x_{i-1}) = f(c_i)\Delta x.$$

Then, substituting into the previous equation, we have

$$F(b) - F(a) = \sum_{i=1}^{n} f(c_i)\Delta x.$$

Taking the limit of both sides as $n \to \infty$, we obtain

$$F(b) - F(a) = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x = \int_a^b f(x)\,dx.$$

Evaluating an Integral with the Fundamental Theorem of Calculus

Problem: Evaluate $\int_{-2}^{2}(t^2 - 4)\,dt$.
Solution: Using the Fundamental Theorem of Calculus, we find the antiderivative and evaluate at the limits:

$$\begin{aligned}\int_{-2}^{2}(t^2-4)\,dt &= \left(\frac{t^3}{3} - 4t\right)\Big|_{-2}^{2} \\ &= \left[\frac{2^3}{3} - 4(2)\right] - \left[\frac{(-2)^3}{3} - 4(-2)\right] \\ &= \left(\frac{8}{3} - 8\right) - \left(-\frac{8}{3} + 8\right) \\ &= \frac{8}{3} - 8 + \frac{8}{3} - 8 \\ &= \frac{16}{3} - 16 = -\frac{32}{3}.\end{aligned}$$

Evaluating a Definite Integral Using the Fundamental Theorem of Calculus, Part 2

Problem: Evaluate $\int_1^9 \frac{x-1}{\sqrt{x}}\, dx$ using the Fundamental Theorem of Calculus, Part 2.

Solution: First, eliminate the radical by rewriting the integral using rational exponents. Then, separate the numerator terms:

$$\int_1^9 \frac{x-1}{x^{1/2}}\, dx = \int_1^9 \left(x^{1/2} - x^{-1/2}\right)\, dx.$$

Now, integrate using the power rule for antiderivatives:

$$\begin{aligned}\int_1^9 \left(x^{1/2} - x^{-1/2}\right)\, dx &= \left(\frac{x^{3/2}}{\frac{3}{2}} - \frac{x^{1/2}}{\frac{1}{2}}\right)\Bigg|_1^9 \\ &= \left[\frac{2}{3}(27) - 2(3)\right] - \left[\frac{2}{3}(1) - 2(1)\right] \\ &= 18 - 6 - \frac{2}{3} + 2 = \frac{40}{3}.\end{aligned}$$

Evaluate $\int_1^2 x^{-4}\,dx$

Problem: Evaluate the definite integral $\int_1^2 x^{-4}\,dx$.

Solution: To find the antiderivative, use the power rule for integration:

$$\begin{aligned} \int x^{-4}\,dx &= \frac{x^{-3}}{-3} + C \\ &= -\frac{1}{3x^3} + C. \end{aligned}$$

Now, apply the Fundamental Theorem of Calculus:

$$\begin{aligned} \int_1^2 x^{-4}\,dx &= \left[-\frac{1}{3x^3}\right]_1^2 \\ &= \left(-\frac{1}{3(2)^3}\right) - \left(-\frac{1}{3(1)^3}\right) \\ &= -\frac{1}{24} + \frac{1}{3} = \frac{7}{24}. \end{aligned}$$

Answer: $\int_1^2 x^{-4}\,dx = \frac{7}{24}$.

Roller-Skating Race: James vs. Kathy

James's Velocity: $f(t) = 5 + 2t$ ft/sec
To find James's total distance traveled, integrate $f(t)$ over the interval $[0, 5]$:

$$\int_0^5 (5 + 2t)\, dt = \left[5t + \frac{1}{2}t^2\right]_0^5$$
$$= (25 + 25) = 50\,\text{ft}.$$

So, James has skated 50 ft after 5 seconds.

Roller-Skating Race: James vs. Kathy

Kathy's Velocity: $g(t) = 10 + \cos(t)\,\text{ft/sec}$
To find Kathy's total distance traveled, integrate $g(t)$ over the interval $[0, 5]$:

$$\begin{aligned}\int_0^5 (10 + \cos(t))\,dt &= [10t + \sin(t)]_0^5 \\ &= (50 + \sin(5)) - (0 - \sin 0) \\ &= 50 + \sin(5).\end{aligned}$$

Since $\pi < 5 < 2\pi$, $\sin(5) < 0$. Therefore, Kathy has skated a bit less than 50 ft after 5 seconds. James wins, but not by much!

Rematch: James vs. Kathy

Suppose James and Kathy have a rematch, but this time the contest is stopped after only 3 seconds. Let's evaluate the distances covered:

James's Velocity: $f(t) = 5 + 2t$ ft/sec

To find James's total distance in 3 seconds:

$$\int_0^3 (5 + 2t)\,dt = \left[5t + \frac{1}{2}t^2\right]_0^3$$
$$= (15 + \frac{9}{2}) = 24\,\text{ft}.$$

Kathy's Velocity: $g(t) = 10 + \cos(t)$ ft/sec

To find Kathy's total distance in 3 seconds:

$$\int_0^3 (10 + \cos(t))\,dt = [10t + \sin(t)]_0^3$$
$$= (30 + \sin(3)) - (0 - \sin 0)$$
$$= 30 + \sin(3).$$

1.5 Substitution

January 30, 2024

Outline

Learning Objectives

- Use substitution to evaluate indefinite integrals.
- Use substitution to evaluate definite integrals.

Substitution for Indefinite Integrals

Let $u = g(x)$, where $g'(x)$ is continuous, let $f(x)$ be continuous over the range of g, and let $F(x)$ be an antiderivative of $f(x)$. Then,

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du = F(u) + C = F(g(x)) + C.$$

Proof

Let f, g, u, and F be as specified in the theorem. Then

$$\frac{d}{dx}\Big(F(g(x))\Big) = F'(g(x))g'(x) = f(g(x))g'(x).$$

This means that $F(g(x))$ is an antiderivative of $f(g(x))g'(x)$ and hence

$$\int f(g(x))g'(x)\,dx = F(g(x)) + C.$$

Since $u = g(x)$ and F is an antiderivative of f, we have that $F(g(x)) + C = F(u) + C = \int f(u)\,du$, which completes the proof. $\square$

Example: Substitution for Indefinite Integrals

Returning to the problem we looked at originally, we let $u = x^2 - 3$ and then $du = 2x\,dx$. Rewriting the integral in terms of u, we obtain:

$$\int \underbrace{(x^2 - 3)}_{u^3} \underbrace{(2x\,dx)}_{du} = \int u^3\,du.$$

Using the power rule for integrals, we have:

$$\int u^3\,du = \frac{u^4}{4} + C.$$

Substituting the original expression for x back into the solution, we get:

$$\frac{u^4}{4} + C = \frac{(x^2 - 3)^4}{4} + C.$$

Problem-Solving Strategy: Integration by Substitution

Integration by Substitution

1. Look carefully at the integrand and select an expression $g(x)$ within the integrand to set equal to u. Quite often, we select $g(x)$ so that $g'(x)$ is also part of the integrand.
2. Substitute $u = g(x)$ and $du = g'(x)\,dx$ into the integral.
3. We should now be able to evaluate the integral with respect to u. If the integral can't be evaluated, we need to go back and select a different expression to use as u.
4. Evaluate the integral in terms of u.
5. Replace u with $g(x)$ to write the result in terms of x.

Using Substitution to Evaluate an Indefinite Integral

Problem: Evaluate $\int 6x(3x^2+4)^4\,dx$.
Solution:

1. Choose $u = 3x^2 + 4$, so $du = 6x\,dx$.
2. Write the integral in terms of u:
$$\int 6x(3x^2+4)^4\,dx = \int u^4\,du.$$
3. Evaluate the integral with respect to u and then return to the variable x:
$$\int u^4\,du = \frac{u^5}{5} + C = \frac{(3x^2+4)^5}{5} + C.$$

Analysis: The derivative of the result of integration confirms the correctness of our answer.

Using Substitution to Evaluate an Indefinite Integral

Problem: Evaluate $\int 3x^2(x^3-3)^2\,dx$.

Solution:

1. Choose $u = x^3 - 3$, so $du = 3x^2\,dx$.
2. Write the integral in terms of u:

$$\int 3x^2(x^3-3)^2\,dx = \int u^2\,du.$$

3. Evaluate the integral with respect to u and then return to the variable x:

$$\int u^2\,du = \frac{u^3}{3} + C = \frac{(x^3-3)^3}{3} + C.$$

Answer:

$$\int 3x^2(x^3-3)^2\,dx = \frac{1}{3}(x^3-3)^3 + C.$$

Using Substitution with Alteration

Problem: Evaluate $\int z\sqrt{z^2-5}\,dz$. **Solution:**

1. Let $u = z^2 - 5$ and $du = 2z\,dz$. To match the integrand, multiply both sides of the du equation by $\frac{1}{2}$:
$$\frac{1}{2}du = z\,dz.$$
2. Rewrite the integral in terms of u:
$$\int z\sqrt{z^2-5}\,dz = \int \sqrt{u}\cdot\frac{1}{2}\,du = \frac{1}{2}\int\sqrt{u}\,du.$$
3. Integrate the expression in u using the power rule:
$$\frac{1}{2}\int\sqrt{u}\,du = \frac{1}{2}\left(\frac{2}{3}\right)u^{3/2} + C = \frac{1}{3}u^{3/2} + C = \frac{1}{3}(z^2-5)^{3/2} + C.$$

Answer:
$$\int z\sqrt{z^2-5}\,dz = \frac{1}{3}(z^2-5)^{3/2} + C.$$

Use Substitution for Another Integral

Problem: Find the antiderivative of $\int x^2(x^3+5)^9\,dx$.
Hint: Multiply the *du* equation by $\frac{1}{3}$.
Solution:

1. Let $u = x^3 + 5$ and $du = 3x^2\,dx$. To match the integrand, multiply both sides of the *du* equation by $\frac{1}{3}$:
$$\frac{1}{3}du = x^2\,dx.$$
2. Rewrite the integral in terms of u:
$$\int x^2(x^3+5)^9\,dx = \int u^9 \cdot \frac{1}{3}\,du.$$
3. Integrate the expression in u using the power rule:
$$\int u^9 \cdot \frac{1}{3}\,du = \frac{1}{3} \cdot \frac{u^{10}}{10} + C = \frac{1}{30}u^{10} + C.$$
4. Substitute back $u = x^3 + 5$ to obtain the final antiderivative:

Using Substitution with Integrals of Trigonometric Functions

Problem: Evaluate the integral $\int \frac{\sin(t)}{\cos^3(t)}\,dt$. **Solution:**

1. Rewrite the integral as $\int \frac{1}{\cos^3(t)} \cdot \sin(t)\,dt$.
2. Let $u = \cos(t)$. Then, $du = -\sin(t)\,dt$, so $\sin(t)\,dt = -du$.
3. Substitute $-du$ for $\sin(t)\,dt$ and u for $\cos(t)$:
$$\int \frac{\sin(t)}{\cos^3(t)}\,dt = -\int \frac{1}{u^3}\,du.$$
4. Evaluate the integral in terms of u:
$$-\int \frac{1}{u^3}\,du = -\left(-\frac{1}{2}\right)u^{-2} + C = \frac{1}{2}u^{-2} + C.$$
5. Substitute $u = \cos(t)$ back into the expression:
$$\frac{1}{2}\cos^{-2}(t) + C = \frac{1}{2\cos^2(t)} + C.$$

Using Substitution with Trigonometric Functions

Problem: Evaluate the integral $\int \cos(t) \cdot 2^{\sin(t)}\, dt$.

Solution:

1. Let $u = \sin(t)$. Then, $du = \cos(t)\, dt$.
2. Substitute u and du into the integral:
$$\int \cos(t) \cdot 2^{\sin(t)}\, dt = \int 2^u\, du.$$
3. Evaluate the integral with respect to u:
$$\int 2^u\, du = \frac{2^u}{\ln(2)} + C.$$
4. Substitute back $u = \sin(t)$:
$$\frac{2^{\sin(t)}}{\ln(2)} + C.$$

Basic Trigonometric Integrals with Substitution

$$\int \tan(x)\,dx = -\ln|\cos(x)| + C = \ln|\sec(x)| + C$$

$$\int \cot(x)\,dx = \ln|\sin(x)| + C$$

$$\int \sec(x)\,dx = \ln|\sec(x) + \tan(x)| + C$$

$$\int \csc(x)\,dx = -\ln|\csc(x) + \cot(x)| + C = \ln|\csc(x) - \cot(x)| + C$$

Evaluating an Indefinite Integral Using Substitution

Problem: Evaluate the integral $\int \frac{x}{\sqrt{x-1}}\, dx$ using substitution. **Solution:** If we let $u = x - 1$, then $du = dx$. But this does not account for the x in the numerator of the integrand. We need to express x in terms of u to complete the substitution. If $u = x - 1$, then $x = u + 1$. Now we can rewrite the integral in terms of u:

$$\begin{aligned}\int \frac{x}{\sqrt{x-1}}\, dx &= \int \frac{u+1}{\sqrt{u}}\, du \\ &= \int \left(\sqrt{u} + \frac{1}{\sqrt{u}}\right)\, du \\ &= \int \left(u^{1/2} + u^{-1/2}\right)\, du.\end{aligned}$$

Then we integrate in the usual way, replace u with the original expression, and factor and simplify the result. Thus,

$$\begin{aligned}\int \left(u^{1/2} + u^{-1/2}\right)\, du &= \frac{2}{3}u^{3/2} + 2u^{1/2} + C \\ &= \frac{2}{3}(x-1)^{3/2} + 2(x-1)^{1/2} + C.\end{aligned}$$

Using Substitution to Evaluate an Indefinite Integral

Problem: Evaluate the indefinite integral $\int t(1-2t)^7\,dt$ using substitution.

Solution: Let $u = 1-2t$. Then, $du = -2\,dt$ or $dt = -\frac{1}{2}\,du$. Substituting $u = 1-2t$ and $dt = -\frac{1}{2}\,du$ into the integral, we have:

$$\begin{aligned}\int t(1-2t)^7\,dt &= -\frac{1}{2}\int t\,du = -\frac{1}{2}\int \frac{u}{-2}\,du = \frac{1}{4}\int u\,du \\ &= \frac{1}{4}\cdot\frac{u^2}{2} + C = \frac{1}{8}u^2 + C = \frac{1}{8}(1-2t)^2 + C \\ &= \frac{(1-2t)^2}{8} + C \\ &= \frac{(1-2t)^9}{36} - \frac{(1-2t)^8}{32} + C.\end{aligned}$$

Substitution for Definite Integrals

Let $u = g(x)$, where $g'(x)$ is continuous over an interval $[a, b]$, and let f be continuous over the range of $u = g(x)$. Then,

$$\int_a^b f(g(x)) \cdot g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du.$$

Using Substitution to Evaluate a Definite Integral

Problem: Evaluate $\int_0^1 (x^3+1)e^{x^4+4x}\,dx$ using substitution.
Solution: Take $u = x^4 + 4x$. Then $du = (4x^3+4)\,dx = 4(x^3+1)\,dx$ and hence $(x^3+1)\,dx = \frac{1}{4}du$. To adjust the bounds of integration, note that $x = 0$ corresponds to $u = 0^4 + 4\cdot 0 = 0$ and $x = 1$ corresponds to $u = 1^4 + 4\cdot 1 = 5$. We then obtain

$$\int_0^1 (x^3+1)e^{x^4+4x}\,dx = \int_0^5 \frac{1}{4}e^u\,du = \frac{1}{4}e^u\Big|_0^5 = \frac{e^5-1}{4}.$$

Using Substitution to Evaluate a Definite Integral

Problem: Evaluate $\int_1^e \frac{\ln(x)}{x}\,dx$ using substitution.
Solution: Take $u = \ln(x)$. Then $du = \frac{1}{x}\,dx$ and the bounds of integration transform as follows: $x = 1 \Rightarrow u = \ln(1) = 0$ and $x = e \Rightarrow u = \ln(e) = 1$. We rewrite the integral in terms of u:

$$\int_1^e \frac{\ln(x)}{x}\,dx = \int_0^1 u\,du.$$

Now, integrating the expression with respect to u, we get:

$$\int_0^1 u\,du = \frac{1}{2}u^2\Big|_0^1 = \frac{1}{2}(1^2 - 0^2) = \frac{1}{2}.$$

Therefore, $\int_1^e \frac{\ln(x)}{x}\,dx = \frac{1}{2}$.

Using Substitution to Evaluate a Definite Integral

Problem: Evaluate $\int_{1/2}^{1} \frac{\sin\left(\frac{1}{x}\right)}{x^2}\, dx$ using substitution.
Solution: Let $u = \frac{1}{x} = x^{-1}$. Then $du = -\frac{1}{x^2}\, dx$ and $x = \frac{1}{2} \Rightarrow u = 2$, and $x = 1 \Rightarrow u = 1$. We rewrite the integral in terms of u:

$$\int_{1/2}^{1} \frac{\sin\left(\frac{1}{x}\right)}{x^2}\, dx = \int_{2}^{1} \sin(u) \cdot (-1) du = (\cos(u))\Big|_{2}^{1} = \cos(1) - \cos(2).$$

Analysis: Note that the lower limit of integration was bigger than the upper limit in the integral in terms of u. This often happens when using substitution, and it's not an issue.
Answer: $\cos(1) - \cos(2)$

Using Substitution to Evaluate a Definite Integral (Cont'd)

Problem: Evaluate $\int_{\pi^2/16}^{\pi^2/9} \frac{\sec^2(\sqrt{x})}{\sqrt{x}}\, dx$ using substitution.

Solution: Take $u = \sqrt{x}$. Then $du = \frac{1}{2\sqrt{x}} dx$ implies that $dx = 2\sqrt{x} du$, $x = \pi^2/16 \Rightarrow u = \pi/4$, and $x = \pi^2/9 \Rightarrow u = \pi/3$. We rewrite the integral in terms of u:

$$\int_{\pi^2/16}^{\pi^2/9} \frac{\sec^2(\sqrt{x})}{\sqrt{x}}\, dx = \int_{1/4}^{1/3} 2\sec^2(u)\, du = 2\tan(u)\Big|_{\pi/4}^{\pi/3}.$$

Answer: $2(\tan(\pi/3) - \tan(\pi/4)) = 2(\sqrt{3} - 1)$

Evaluating a Definite Integral using Substitution

Problem: Evaluate $\int_0^1 x^5(1-x^3)^4\,dx$ using substitution.
Solution: Let $u = 1 - x^3$, then $du = -3x^2dx$. We need to express $x^5\,dx$ in terms of u: $x^5\,dx = (1-u)\left(-\frac{1}{3}\right)du$. Adjusting the limits, $x = 0 \Rightarrow u = 1$, and $x = 1 \Rightarrow u = 0$. We rewrite the integral in terms of u:

$$\int_0^1 x^5(1-x^3)^4\,dx = -\frac{1}{3}\int_1^0 u^4(1-u)du = -\frac{1}{3}\int_1^0 (u^4 - u^5)du$$

$$= \left(-\frac{1}{3}\right)\left(\frac{u^5}{5} - \frac{u^6}{6}\right)\Bigg|_1^0$$

$$= -\frac{1}{3}\left[(0-0) - \left(\frac{1}{5} - \frac{1}{6}\right)\right]$$

$$= \frac{1}{90}.$$

Evaluating a Definite Integral using Substitution (Cont'd)

Problem: Evaluate $\int_{-1}^{0} \frac{y^3}{y^2+1}\, dy$ using substitution.
Solution: Take $u = y^2 + 1$. Then $du = 2ydy$, $y = -1 \Rightarrow u = 2$, and $y = 0 \Rightarrow u = 1$. We rewrite the integral in terms of u:

$$\int_{-1}^{0} \frac{y^3}{y^2+1}\, dy = \int_{0}^{2} ??\, du.$$

Answer: $\frac{\ln(2)-1}{2}$

1.4 The Net Change Theorem and Integrals of Symmetric Functions

January 30, 2024

Outline

Learning Objectives

- Explain the significance of the net change theorem.
- Use the net change theorem to solve applied problems.
- Apply the integrals of odd and even functions.

Net Change Theorem

The new value of a changing quantity equals the initial value plus the integral of the rate of change:

$$F(b) = F(a) + \int_a^b F'(x)\,dx$$

or

$$\int_a^b F'(x)\,dx = F(b) - F(a).$$

Net Change Theorem

The Net Change Theorem

$$\int_a^b F'(t)dt = F(b) - F(a)$$

Rate of change of $F(t)$

Final value –Initial Value

= Net change of $F(t)$ from $t = a$ to $t = b$

The integral of rate of change is the net change

Finding Net Displacement

Given a velocity function $v(t) = 3t - 5$ (in meters per second) for a particle in motion from time $t = 0$ to time $t = 3$, find the net displacement of the particle.

Solution:

Applying the net change theorem, we have

$$\int_0^3 (3t - 5)\, dt$$

$$\int x^n\, dx = \frac{x^{n+1}}{n+1} + C$$

Finding Net Displacement

Given a velocity function $v(t) = 3t - 5$ (in meters per second) for a particle in motion from time $t = 0$ to time $t = 3$, find the net displacement of the particle.

Solution:

Applying the net change theorem, we have

$$\int_0^3 (3t-5)\,dt = \frac{3t^2}{2} - 5t\bigg|_0^3 = \left[\frac{3(3)^2}{2} - 5(3)\right] - 0$$
$$= \frac{27}{2} - 15 = \frac{27}{2} - \frac{30}{2}$$
$$= -\frac{3}{2}.$$

The net displacement is $-\frac{3}{2}$ meters.

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$$

Finding Total Distance Traveled

Given the velocity function $v(t) = 3t - 5$ m/sec over the time interval $[0, 3]$, we want to find the total distance traveled by the particle.
Solution: To find the total distance traveled, we integrate the absolute value of the velocity function:

$$\int_0^3 |v(t)|\,dt = \int_0^{5/3} (-v(t))\,dt + \int_{5/3}^3 v(t)\,dt = \int_0^{5/3} (5-3t)\,dt + \int_{5/3}^3 (3t-5)\,dt$$

$$= \left(5t - \frac{3t^2}{2}\right)\Big|_0^{5/3} + \left(\frac{3t^2}{2} - 5t\right)\Big|_{5/3}^3$$

$$= \left[5\left(\frac{5}{3}\right) - \frac{3\left(\frac{5}{3}\right)^2}{2}\right] + \left[\frac{27}{2} - 15\right] - \left[\frac{3\left(\frac{5}{3}\right)^2}{2} - \frac{25}{3}\right]$$

$$= \frac{25}{3} - \frac{25}{6} + \frac{27}{2} - 15 - \frac{25}{6} + \frac{25}{3} = \frac{41}{6}.$$

So, the total distance traveled is $\frac{41}{6}$ m.

Finding Net Displacement and Total Distance Traveled

Given the velocity function $f(t) = \frac{1}{2}e^t - 2$ over the interval $[0, 2]$, we want to find the net displacement and the total distance traveled by the particle.

Solution:

1. **Net Displacement:** To find the net displacement, we apply the net change theorem:

$$\int_0^2 f(t)\,dt = \left[\frac{1}{2}e^t - 2t\right]\Bigg|_0^2 = \left[\frac{1}{2}e^2 - 4\right] - \left[\frac{1}{2}e^0 - 0\right] = \frac{1}{2}e^2 - 4 - \frac{1}{2}.$$

   So, the net displacement is $\frac{1}{2}e^2 - \frac{9}{2}$ m.

2. **Total Distance Traveled:** To find the total distance traveled, we integrate the absolute value of the velocity function:

$$\int_0^2 |f(t)|\,dt = \int_0^2 \left|\frac{1}{2}e^t - 2\right|\,dt = ??.$$

How Many Gallons of Gasoline Are Consumed?

If the motor on a motorboat is started at $t = 0$ and the boat consumes gasoline at the rate of $(5 - t^3)$ gal/hr, how much gasoline is used in the first 2 hours?

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Solution: Express the problem as a definite integral, integrate, and evaluate using the Fundamental Theorem of Calculus. The limits of integration are the endpoints of the interval $[0, 2]$. We have

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Solution: Express the problem as a definite integral, integrate, and evaluate using the Fundamental Theorem of Calculus. The limits of integration are the endpoints of the interval $[0, 2]$. We have

$$\int_0^2 (5 - t^3)\, dt$$

$$\int x^n\, dx = \frac{x^{n+1}}{n+1} + C$$

How Many Gallons of Gasoline Are Consumed?

If the motor on a motorboat is started at $t = 0$ and the boat consumes gasoline at the rate of $(5 - t^3)$ gal/hr, how much gasoline is used in the first 2 hours?

Solution: Express the problem as a definite integral, integrate, and evaluate using the Fundamental Theorem of Calculus. The limits of integration are the endpoints of the interval $[0, 2]$. We have

$$\int_0^2 (5 - t^3)\, dt = \left(5t - \frac{t^4}{4}\right)\Bigg|_0^2 = \left[5(2) - \frac{(2)^4}{4}\right] - 0 = 10 - \frac{16}{4} = 6.$$

Thus, the motorboat uses 6 gal of gas in 2 hours.

$$\int x^n\, dx = \frac{x^{n+1}}{n+1} + C$$

Chapter Opener: Iceboats

Figure: Iceboat in action. (Credit: modification of work by Carter Brown, Flickr)

Andrew sets out. As he prepares his iceboat, the wind intensifies. During the first half-hour, the wind speed increases according to:

$$v(t) = \begin{cases} 20t + 5 & \text{for } 0 \leq t \leq \frac{1}{2} \\ 15 & \text{for } \frac{1}{2} \leq t \leq 1 \end{cases}$$

Recalling that Andrew's iceboat travels at twice the wind speed, and assuming he moves in a straight line away from his starting point, how far is Andrew from his starting point after 1 hour?

Solution

To figure out how far Andrew has traveled, we need to integrate his velocity, which is twice the wind speed. Then

$$\text{Distance} = \int_0^1 2v(t)\,dt.$$

$$\begin{aligned}
\int_0^1 2v(t)\,dt &= \int_0^{\frac{1}{2}} 2v(t)\,dt + \int_{\frac{1}{2}}^1 2v(t)\,dt \\
&= \int_0^{\frac{1}{2}} 2(20t+5)\,dt + \int_{\frac{1}{2}}^1 2(15)\,dt = \int_0^{\frac{1}{2}} (40t+10)\,dt + \int_{\frac{1}{2}}^1 30\,dt \\
&= \left[20t^2 + 10t\right]\Big|_0^{\frac{1}{2}} + [30t]\Big|_{\frac{1}{2}}^1 = \left(\frac{20}{4} + 5\right) - 0 + (30-15) = 25.
\end{aligned}$$

So Andrew is 25 miles from his starting point after 1 hour.

Andrew's Iceboating Outing

Suppose that, instead of remaining steady during the second half hour of Andrew's outing, the wind starts to die down according to the function

$$v(t) = \begin{cases} 20t + 5 & \text{for } 0 \le t \le \frac{1}{2} \\ -10t + 15 & \text{for } \frac{1}{2} \le t \le 1 \end{cases}$$

Under these conditions, how far from his starting point is Andrew after 1 hour?

$$\text{Distance} = \int_0^1 2v(t)\, dt.$$

Answer: 17.5 miles.

Integrals of Even and Odd Functions

Suppose that the function f is continuous over the interval $[-a, a]$.

If f is even:

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

If f is odd:

$$\int_{-a}^{a} f(x)\,dx = 0$$

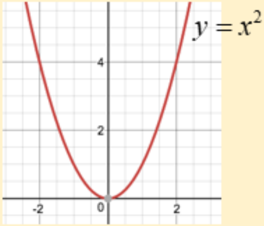
Even and odd functions

Even Functions

$$f(-x) = f(x)$$

Function is unchanged when reflected about the y-axis.

Example:

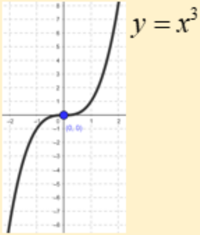


Odd Functions

$$f(-x) = -f(x)$$

Function is unchanged when rotated 180° about the origin.

Example:



Integrating an Even Function

Integrate the even function $\int_{-2}^{2}(3x^8 - 2)\,dx$ and verify that the integration formula for even functions holds.

Solution:

First, we formally verify that the integrand function is even. Let $f(x) = 3x^8 - 2$. Then $f(-x) = 3(-x)^8 - 2 = 3x^8 - 2 = f(x)$, which precisely means that f is even. We then have

$$\int_{-2}^{2}(3x^8 - 2)\,dx = \left(\frac{x^9}{3} - 2x\right)\Bigg|_{-2}^{2}$$

$$= \left[\frac{2^9}{3} - 2(2)\right] - \left[\frac{(-2)^9}{3} - 2(-2)\right]$$

$$= \left(\frac{512}{3} - 4\right) - \left(-\frac{512}{3} + 4\right)$$

$$= \frac{1000}{3}.$$

even functions

To verify the integration formula for even functions, we can calculate the integral from 0 to 2, then double it and check to make sure we get the same answer:

$$\int_0^2 (3x^8 - 2)\, dx = \left(\frac{x^9}{3} - 2x\right)\Bigg|_0^2$$
$$= \frac{512}{3} - 4$$
$$= \frac{500}{3}.$$

Since $2 \times \frac{500}{3} = \frac{1000}{3}$, we have verified the formula for even functions in this particular example.

Integrating an Odd Function

Verify that the function $f(x) = \sin^3(x)(x^2+1)$ is odd and use this fact to evaluate the definite integral $\int_{-5}^{5} f(x)\,dx$.

Solution:

Integrating an Odd Function

Verify that the function $f(x) = \sin^3(x)(x^2 + 1)$ is odd and use this fact to evaluate the definite integral $\int_{-5}^{5} f(x)\,dx$.

Solution: Substituting $-x$ into f, we obtain

$$\begin{aligned} f(-x) &= \sin^3(-x)((-x)^2 + 1) \\ &= (-\sin(x))^3(x^2 + 1) \\ &= -\sin^3(x)(x^2 + 1) \\ &= -f(x), \end{aligned}$$

which proves that f is odd. Because f is continuous over the whole real line as a product of a polynomial and a sine function, it is also continuous over $[-5, 5]$, and we can apply the above result to conclude that $\int_{-5}^{5} \sin^3(x)(x^2 + 1)\,dx = 0$.

Using Properties of Symmetric Functions

Consider the function $f(x) = x^4$. This function is an even function because $f(-x) = (-x)^4 = x^4 = f(x)$ for all x. Since $f(x)$ is even, we can use the property of integrals of symmetric functions:

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

where a is the interval length. Applying this property , we have:

$$\int_{-2}^{2} x^4\,dx = 2\int_{0}^{2} x^4\,dx$$

Now, let's evaluate the integral $\int_0^2 x^4\,dx$:

$$\int_{0}^{2} x^4\,dx = \left[\frac{x^5}{5}\right]_0^2 = \frac{2^5}{5} - \frac{0^5}{5} = \frac{32}{5}$$

Finally, multiply by 2 to get the value of $\int_{-2}^{2} x^4\,dx = 2 \times \frac{32}{5} = \frac{64}{5}$

Quiz: Integrating Even and Odd Functions

Problem 1: Determine if the following functions are even, odd, or neither:

1. $f(x) = x^3 + 2$
2. $g(x) = \sin(x) + \cos(x)$
3. $h(x) = e^x + e^{-x}$

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Solution Problem 1:

1. $f(x)$: Neither (not symmetric about the y-axis)
2. $g(x)$: Neither (not symmetric about the origin)
3. $h(x)$: Even (symmetric about the y-axis)

Quiz: Is $f(x)$ an odd function, even function, or neither? s

Problem 1: Consider the function $f(x) = x^3 - x^2 + 4x + 2$.

Problem 2: Consider the function $f(x) = -x^2 + 10$.

Problem 3: Consider the function $f(x) = x^3 + 4x$.

Problem 4: Consider the function $f(x) = -x^3 + 5x - 2$.

Problem 5: Consider the function $f(x) = \sqrt{x^4} - x^2 + 4$.

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