# Ralated Rates

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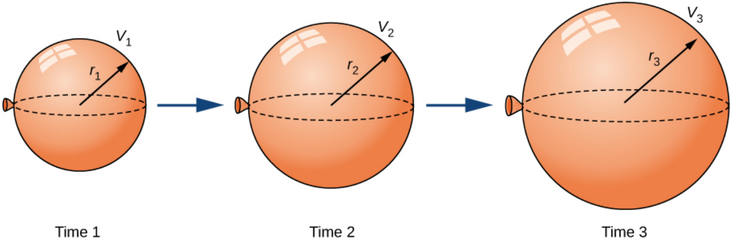
# Learning Objectives

- Express changing quantities in terms of derivatives.
- Find relationships among derivatives in a given problem.
- Use the chain rule to find the rate of change of one quantity based on the rate of change of other quantities.

# Application

In many real-world applications, related quantities are changing with respect to time. For example, if we consider the balloon example again, we can say that the rate of change in the volume, $V$, is related to the rate of change in the radius, $r$. In this case, we say that $\frac{dV}{dt}$ and $\frac{dr}{dt}$ are **related rates** because $V$ is related to $r$.



# Steps

## Step 1: Volume of a Sphere

The volume of a sphere of radius $r$ centimeters is:

$$V = \frac{4}{3}\pi r^3 \,\text{cm}^3.$$

Since the balloon is being filled with air, both the volume and the radius are functions of time. Therefore, $t$ seconds after beginning to fill the balloon with air, the volume of air in the balloon is:

$$V(t) = \frac{4}{3}\pi \left[r(t)\right]^3 \ \text{cm}^3.$$

## Step 2: Differentiate and Apply Chain Rule

Differentiating both sides of the equation with respect to time $t$, and applying the chain rule, we get:

$$\frac{dV}{dt} = 4\pi \left[r(t)\right]^2 \cdot \frac{dr}{dt}.$$

This equation shows that the rate of change of the volume $\frac{dV}{dt}$ is related to the rate of change of the radius $\frac{dr}{dt}$.

## Step 3: Known Rate of Volume Change

The balloon is being filled with air at the constant rate of 2 $\text{cm}^3$/sec, so:

$$\frac{dV}{dt} = 2\,\text{cm}^3/\text{sec}.$$

Substituting into the equation, we have:

$$2 = 4\pi \left[r(t)\right]^2 \cdot \frac{dr}{dt} \textbf{ Then } \frac{dr}{dt} = \frac{1}{2\pi \left[r(t)\right]^2}\,\text{cm}/\text{sec}.$$

# Step 4: Substitute r = 3 cm

When the radius $r = 3\,\text{cm}$, substituting this into the equation gives:

$$\frac{dr}{dt} = \frac{1}{18\pi}\,\text{cm/sec}.$$

Therefore, the radius of the balloon is increasing at a rate of $\frac{1}{18\pi}$ cm/sec.

# Problem-Solving Strategy: Solving a Related-Rates

To solve a related-rates problem, follow these steps:

1. **Assign symbols** to all variables involved in the problem. Draw a figure if applicable.
2. **State the information** given in terms of the variables and identify the rate that needs to be determined.
3. **Find an equation** relating the variables from step 1.
4. **Differentiate** both sides of the equation from step 3 with respect to the independent variable, using the chain rule. This will give an equation relating the derivatives.
5. **Substitute known values** into the equation from step 4 and solve for the unknown rate of change.

**Remember**

Remember not to substitute values too soon, as this could turn a variable into a constant prematurely.

# Problem: Airplane Flying Overhead

An airplane is flying at a constant height of 4000 ft. A man is viewing the plane from a position 3000 ft from the base of a radio tower. The airplane is flying horizontally away from the man at 600 ft/sec.

**Question:** At what rate is the distance between the man and the plane increasing when the plane passes over the radio tower?

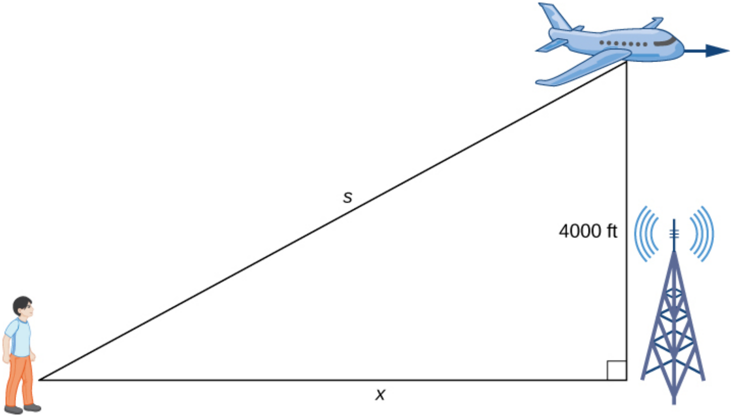
## Step 1: Assign Variables

**Let:**

- $x(t)$ = the horizontal distance between the man and the point on the ground directly below the airplane (changing with time),
- $s(t)$ = the slant distance between the man and the airplane,
- Height of the plane is constant at 4000 ft.

We are tasked to find $\frac{ds}{dt}$ when $x(t) = 3000$ ft.

# Graph



## Step 2: Write Known Information

The horizontal distance is increasing at a constant rate:

$$\frac{dx}{dt} = 600\,\text{ft/sec}.$$

We need to find the rate of change of the distance between the man and the plane, i.e., $\frac{ds}{dt}$, when $x = 3000\,\text{ft}$.

## Step 3: Relating the Variables

From the geometry of the problem (right triangle), we can relate $x(t)$ and $s(t)$ using the Pythagorean theorem:

$$s(t)^2 = x(t)^2 + 4000^2.$$

Differentiating $2s(t)\frac{ds}{dt} = 2x(t)\frac{dx}{dt}$. Simplifying:

$$s(t)\frac{ds}{dt} = x(t)\frac{dx}{dt}.$$

# Steps 4: Solve for $\frac{ds}{dt}$

Solving for $\frac{ds}{dt}$, we get:

$$\frac{ds}{dt} = \frac{x(t)\frac{dx}{dt}}{s(t)}.$$

When $x = 3000\,\text{ft}$, we find $s$ using the Pythagorean theorem:

$$s = \sqrt{3000^2 + 4000^2} = 5000\,\text{ft}.$$

Substituting the known values:

$$\frac{ds}{dt} = \frac{3000 \times 600}{5000} = 360\,\text{ft}/\text{sec}.$$

**Answer:** The distance between the man and the airplane is increasing at 360 ft/sec.

# Airplane

An airplane is flying at a constant height of 4000 ft. A man is viewing the plane from a position 3000 ft from the base of a radio tower. The airplane is flying horizontally away from the man at 600 ft/sec.

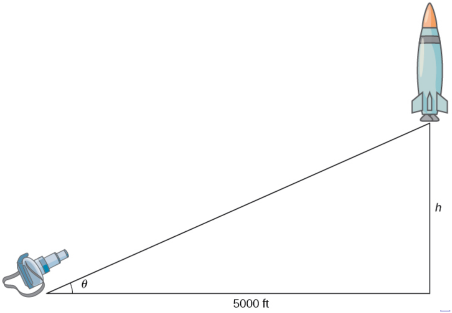
**Question:** At what rate is the distance between the man and the plane increasing when the plane passes over the radio tower?

# Step 1: Assign Variables

Let $h$ denote the height of the rocket above the launch pad and $\theta$ be the angle between the camera lens and the ground.

**Objective:** We are trying to find $\frac{d\theta}{dt}$ when the rocket is 1000 ft above the ground.
**Given:** The rocket is moving at a rate of:

$$\frac{dh}{dt} = 600 \text{ ft/sec.}$$



# Step 2: Relating Variables with Trigonometry

**Relating Variables:**

The right triangle formed by the rocket's height, the distance from the camera to the launch pad, and the hypotenuse helps relate $h$ and $\theta$.

From trigonometry, we know:

$$\tan\theta = \frac{h}{5000}.$$

**This gives us the equation:**

$$h = 5000\tan\theta.$$

# Step 3: Differentiate with Respect to Time

Differentiating both sides of the equation $h = 5000 \tan\theta$ with respect to time $t$:

$$\frac{dh}{dt} = 5000 \sec^2\theta \cdot \frac{d\theta}{dt}.$$

We want to find $\frac{d\theta}{dt}$ when $h = 1000\,\text{ft}$.

So, we need to calculate $\sec^2\theta$ at that point.

# Step 4: Determine $\sec^2\theta$ and Hypotenuse

We know the adjacent side is 5000 ft, and the opposite side is $h = 1000$ ft.

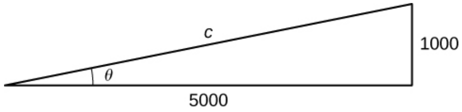
Using the Pythagorean theorem, the hypotenuse $c$ is:

$$c = \sqrt{5000^2 + 1000^2} = 1000\sqrt{26}\,\text{ft}.$$

Therefore:

$$\sec^2\theta = \left(\frac{1000\sqrt{26}}{5000}\right)^2 = \frac{26}{25}.$$

# Step 4: Triangle Diagram




The triangle shows $h = 1000$ ft, adjacent side 5000 ft, and hypotenuse $c = 1000\sqrt{26}$ ft.

# Step 5: Solve for $\frac{d\theta}{dt}$

From Step 3, we have the equation:

$$\frac{dh}{dt} = 5000 \sec^2\theta \cdot \frac{d\theta}{dt}.$$

Substituting the known values $\frac{dh}{dt} = 600\,\text{ft/sec}$ and $\sec^2\theta = \frac{26}{25}$:

$$600 = 5000 \cdot \frac{26}{25} \cdot \frac{d\theta}{dt}.$$

Solving for $\frac{d\theta}{dt}$:

$$\frac{d\theta}{dt} = \frac{3}{26}\,\text{rad/sec}.$$

# Problem: Rate of Change for Camera Angle

**Problem:**
What rate of change is necessary for the elevation angle of the camera if the camera is placed on the ground at a distance of 4000 ft from the launch pad and the velocity of the rocket is 500 ft/sec when the rocket is 2000 ft off the ground?

**Hint:**
Find $\frac{d\theta}{dt}$ when $h = 2000\,\text{ft}$. At that time, we know:

$$\frac{dh}{dt} = 500\,\text{ft/sec}.$$

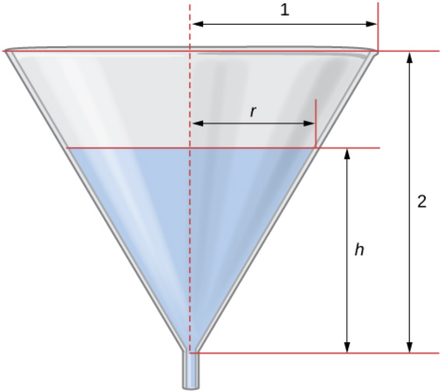
# Solution

**Solution:**
The rate of change of the camera's elevation angle is:

$$\frac{1}{10}\,\text{rad/sec.}$$

# Water Draining from a Funnel

Water is draining from the bottom of a cone-shaped funnel at the rate of $0.03\,\text{ft}^3/\text{sec}$. The height of the funnel is $2\,\text{ft}$ and the radius at the top of the funnel is $1\,\text{ft}$.
At what rate is the height of the water in the funnel changing when the height of the water is $\frac{1}{2}\,\text{ft}$?



# Step 2: Determine $\frac{dh}{dt}$

Let $h$ denote the height of the water in the funnel, $r$ denote the radius of the water at its surface, and $V$ denote the volume of the water.

We need to determine $\frac{dh}{dt}$ when $h = \frac{1}{2}$ ft. We know that:

$$\frac{dV}{dt} = -0.03\,\text{ft}^3/\text{sec}.$$

# Step 3: Volume of Water in the Cone

The volume of water in the cone is given by:

$$V = \frac{1}{3}\pi r^2 h.$$

From the figure, we know that we have similar triangles. Therefore, the ratio of the sides in the two triangles is the same:

$$\frac{r}{h} = \frac{1}{2} \quad \text{or} \quad r = \frac{h}{2}.$$

Using this, the equation for volume becomes:

$$V = \frac{1}{3}\pi \left(\frac{h}{2}\right)^2 h = \frac{\pi}{12} h^3.$$

# Step 4: Apply Chain Rule

Applying the chain rule, we differentiate both sides of the equation with respect to time $t$:

$$\frac{dV}{dt} = \frac{\pi}{4} h^2 \frac{dh}{dt}.$$

# Step 5: Solve for $\frac{dh}{dt}$

We want to find $\frac{dh}{dt}$ when $h = \frac{1}{2}$ ft. Since water is leaving at the rate of $0.03\,\text{ft}^3/\text{sec}$, we know:

$$\frac{dV}{dt} = -0.03\,\text{ft}^3/\text{sec}.$$

Therefore:

$$-0.03 = \frac{\pi}{4}\left(\frac{1}{2}\right)^2 \frac{dh}{dt},$$

which simplifies to:

$$-0.03 = \frac{\pi}{16}\frac{dh}{dt}.$$

# Final Solution

Solving for $\frac{dh}{dt}$, we get:

$$\frac{dh}{dt} = \frac{-0.48}{\pi} \approx -0.153\,\text{ft}/\text{sec}.$$

# Problem: Water Level Rate of Change

**Problem:**

At what rate is the height of the water changing when the height of the water is $\frac{1}{4}$ ft?

**Hint:**

We need to find $\frac{dh}{dt}$ when $h = \frac{1}{4}$.

## Step 1: Volume Formula

The volume of water in the cone is:

$$V = \frac{1}{3}\pi r^2 h.$$

Since $\frac{r}{h} = \frac{1}{2}$, we have $r = \frac{h}{2}$. Substituting this into the volume equation:

$$V = \frac{1}{3}\pi \left(\frac{h}{2}\right)^2 h = \frac{\pi}{12} h^3.$$

## Step 2: Differentiate the Volume

Differentiating both sides of the volume equation with respect to time:

$$\frac{dV}{dt} = \frac{\pi}{4} h^2 \frac{dh}{dt}.$$

## Step 3: Solve for $\frac{dh}{dt}$

We know that water is draining at a rate of $\frac{dV}{dt} = -0.03\,\text{ft}^3/\text{sec}$. Thus:

$$-0.03 = \frac{\pi}{4} \left(\frac{1}{4}\right)^2 \frac{dh}{dt}.$$

Simplifying the equation:

$$-0.03 = \frac{\pi}{4} \times \frac{1}{16} \frac{dh}{dt} = \frac{\pi}{64} \frac{dh}{dt}.$$

Therefore, solving for $\frac{dh}{dt}$:

$$\frac{dh}{dt} = \frac{-0.03 \times 64}{\pi} = \frac{-1.92}{\pi} \approx -0.611\,\text{ft/sec}.$$

# Step 4

The height of the water is decreasing at a rate of approximately:

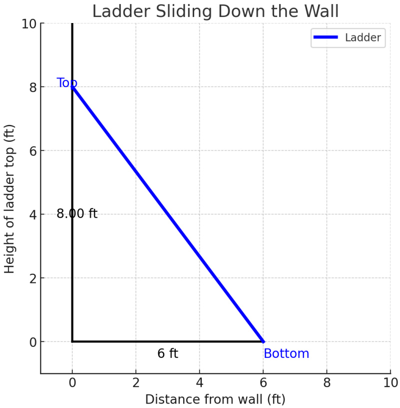
$$\frac{dh}{dt} \approx -0.611\,\text{ft}/\text{sec}$$

when the height of the water is $\frac{1}{4}$ ft.

# Problem: Ladder Sliding Down a Wall

A ladder 10 feet long is leaning against a vertical wall. The bottom of the ladder is pulled away from the wall at a rate of 2 feet per second.

**Question:** How fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 feet away from the wall?



# Problem: Ladder Sliding Down a Wall

- **Step 1: Assign Variables**
- **Step 2: Use the Pythagorean Theorem**
  The relationship between $x$, $y$, and $L$ is given by:
- **Step 3: Differentiate with Respect to Time** Differentiate both sides of the equation with respect to time $t$:
- **Step 4: Solve for $\frac{dy}{dt}$**
- **Step 5: Find $y$ When $x = 6$**
  Use the Pythagorean theorem to find $y$ when $x = 6\,\text{ft}$:
- **Step 6: Final Substitution**
  Substitute $x = 6$, $y$, and $\frac{dx}{dt} = 2\,\text{ft/sec}$ into the equation to find $\frac{dy}{dt}$.

# Key Concepts

To solve a related rates problem:

1. First, **draw a picture** that illustrates the relationship between the two or more related quantities that are changing with respect to time.
2. In terms of the quantities, **state the information** given and the rate to be found.
3. **Find an equation** relating the quantities.
4. **Use differentiation**, applying the chain rule as necessary, to find an equation that relates the rates.
5. Be sure **not to substitute** a variable quantity for one of the variables until after finding an equation relating the rates.

# Maxima and Minima

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# Learning Objectives

## Learning Objectives

- Define absolute extrema.
- Define local extrema.
- Explain how to find the critical numbers of a function over a closed interval.
- Describe how to use critical numbers to locate absolute extrema over a closed interval.

# Practical Significance of Extrema

## Important Note

We are often interested in determining the largest and smallest values of a function. This information is important for:

- Creating accurate graphs.
- Solving optimization problems such as maximizing profit or minimizing material usage.

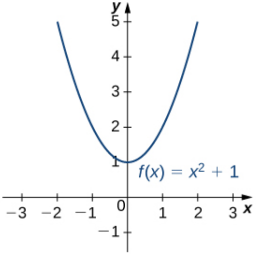
# Absolute Extrema Example

## Example

Consider the function $f(x) = x^2 + 1$ over the interval $(-\infty, \infty)$:

- As $x \to \pm\infty$, $f(x) \to \infty$.
- Therefore, the function does not have a largest value.
- However, $f(x) \geq 1$ for all $x$, and $f(0) = 1$.

**Conclusion:** The function has an **absolute minimum** of 1 at $x = 0$, but no absolute maximum.

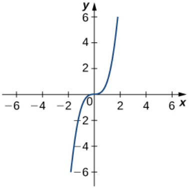


# Definition of Absolute Extrema

**Definition**

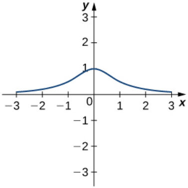
Let $f$ be a function defined over an interval $I$, and let $c \in I$.

- $f$ has an **absolute maximum** on $I$ at $c$ if $f(c) \geq f(x)$ for all $x \in I$.
- $f$ has an **absolute minimum** on $I$ at $c$ if $f(c) \leq f(x)$ for all $x \in I$.
- If $f$ has an absolute maximum or minimum, we say that $f$ has an **absolute extremum** at $c$.



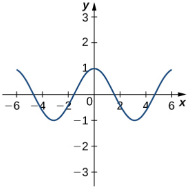

$f(x) = x^3$ on $(-\infty, \infty)$
No absolute maximum
No absolute minimum

(a)



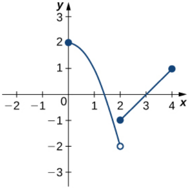

$f(x) = \frac{1}{x^2+1}$ on $(-\infty, \infty)$
Absolute maximum of 1 at $x = 0$
No absolute minimum

(b)



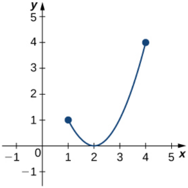

$f(x) = \cos(x)$ on $(-\infty, \infty)$
Absolute maximum of 1 at $x = 0$, $\pm 2\pi$, $\pm 4\pi$...
Absolute minimum of $-1$ at $x = \pm\pi$, $\pm 3\pi$...

(c)



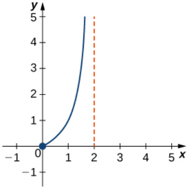

$f(x) = \begin{cases} 2 - x^2 & 0 \le x < 2 \\ x - 3 & 2 \le x \le 4 \end{cases}$
Absolute maximum of 2 at $x = 0$
No absolute minimum

(d)




$f(x) = (x-2)^2$ on $[1, 4]$
Absolute maximum of 4 at $x = 4$
Absolute minimum of 0 at $x = 2$

(e)




$f(x) = \frac{x}{2-x}$ on $[0, 2)$
No absolute maximum
Absolute minimum of 0 at $x = 0$

(f)

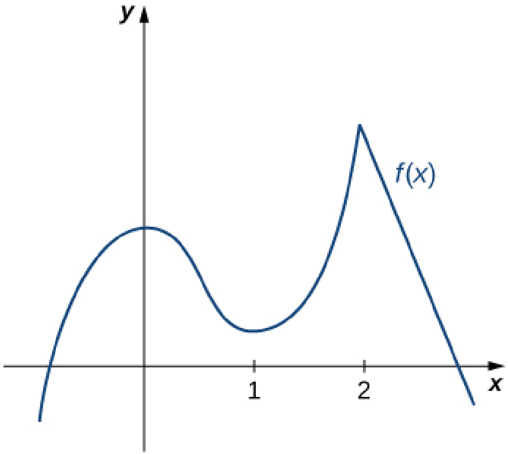
# Extreme Value Theorem

**Theorem**

If $f$ is a continuous function over the closed, bounded interval $[a, b]$, then:

- There is a point in $[a, b]$ at which $f$ has an **absolute maximum**.
- There is a point in $[a, b]$ at which $f$ has an **absolute minimum**.

# Illustration local extrema




$f(x)$ defined on $(-\infty, \infty)$
Local maxima at $x = 0$ and $x = 2$
Local minimum at $x = 1$

# Local Extrema

**A function $f$ has a local maximum at $c$ if**:

- There exists an open interval $I$ containing $c$.
- $I$ is contained in the domain of $f$.
- $f(c) \geq f(x)$ for all $x \in I$.

**A function $f$ has a local minimum at $c$ if**:

- There exists an open interval $I$ containing $c$.
- $I$ is contained in the domain of $f$.
- $f(c) \leq f(x)$ for all $x \in I$.

**A function $f$ has a local extremum at $c$ if**:

- $f$ has a local maximum at $c$, or
- $f$ has a local minimum at $c$.

# Definition and Fermat's Theorem

## Definition

Let $c$ be an interior point in the domain of $f$. We say that $c$ is a **critical number** of $f$ if

$$f'(c) = 0 \quad \text{or} \quad f'(c) \text{ is undefined.}$$

## Fermat's Theorem

If $f$ has a local extremum at $c$ and $f$ is differentiable at $c$, then

$$f'(c) = 0.$$

## Important

Note this theorem does not claim that a function $f$ must have a local extremum at a critical number. Rather, it states that critical numbers are candidates for local extrema.

# Proof Fermat's Theorem

Suppose $f$ has a local extremum at $c$ and $f$ is differentiable at $c$. We need to show that $f'(c) = 0$. To do this, we will show that $f'(c) \geq 0$ and $f'(c) \leq 0$, and therefore $f'(c) = 0$. Since $f$ has a local extremum at $c$, $f$ has a local maximum or local minimum at $c$.

Suppose $f$ has a local maximum at $c$. The case in which $f$ has a local minimum at $c$ can be handled similarly. There then exists an open interval $I$ such that $f(c) \geq f(x)$ for all $x \in I$. Since $f$ is differentiable at $c$, from the definition of the derivative, we know that

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$

Since this limit exists, both one-sided limits also exist and equal $f'(c)$. Therefore,

$$f'(c) = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c}, \textbf{ and } f'(c) = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c}.$$

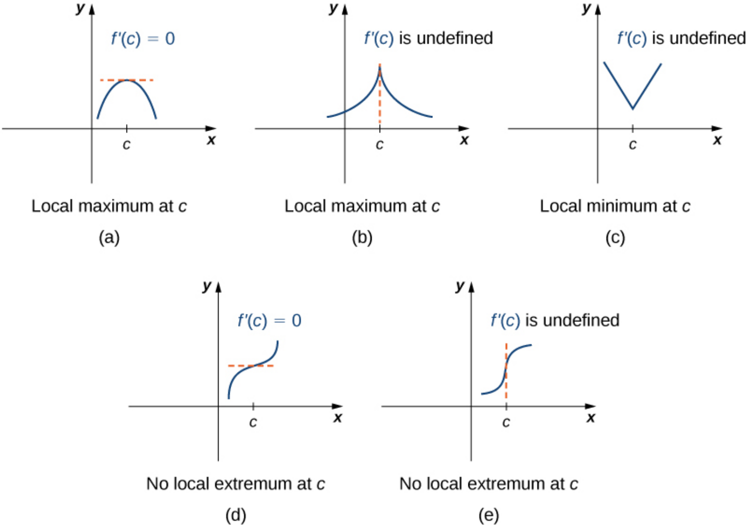
Since $f(c)$ is a local maximum, we see that $f(x) - f(c) \leq 0$ for $x$ near $c$. Therefore, for $x$ near $c$, but $x \neq c$, we have

$$\frac{f(x) - f(c)}{x - c} \leq 0.$$

From this, we conclude that $f'(c) \leq 0$. Similarly, it can be shown that $f'(c) \geq 0$. Therefore,

$$f'(c) = 0.$$

# Defined and undefined $f'(c)$



# Steps to Find Local Minima and Maxima

## Step 1: Find the derivative $f'(x)$

## Step 2: Set the derivative equal to zero: Solve $f'(x) = 0$ **critical numbers**

## Step 3: Determine where the derivative is undefined

## Step 4: Use the First Derivative Test

For each critical point $c$:

- If $f'(x)$ changes from positive to negative at $c$, $f(c)$ is a **local maximum**.
- If $f'(x)$ changes from negative to positive at $c$, $f(c)$ is a **local minimum**.
- If $f'(x)$ does not change sign, $f(c)$ is neither.

# Optional and applicable steps

## Step 5: Use the Second Derivative Test (optional)

If the second derivative $f''(x)$ exists:

- If $f''(c) > 0$, $f(c)$ is a local minimum.
- If $f''(c) < 0$, $f(c)$ is a local maximum.
- If $f''(c) = 0$, the test is inconclusive.

## Step 6: Check the endpoints (if applicable)

If the function is defined on a closed interval, check the function values at the endpoints. These may give the absolute maximum or minimum over the interval.

# Locating Critical Numbers: Example a

For the function $f(x) = \frac{1}{3}x^3 - \frac{5}{2}x^2 + 4x$:

- **Step 1:** Find the derivative $f'(x)$.

$$f'(x) = \_\_\_\_\_\_\_\_$$

- **Step 2:** Set $f'(x) = 0$ and solve for $x$ to find the critical numbers.

$$f'(x) = 0 \quad \Rightarrow \quad x = \_\_\_\_\_\_\_\_$$

- **Step 3:** Determine if there is a local extremum at each critical number using a graphing utility.

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# Locating Critical Numbers: Example b

For the function $f(x) = (x^2 - 1)^3$:

- **Step 1:** Find the derivative $f'(x)$.

$$f'(x) = \_\_\_\_\_\_\_\_$$

- **Step 2:** Set $f'(x) = 0$ and solve for $x$ to find the critical numbers.

$$f'(x) = 0 \quad \Rightarrow \quad x = \_\_\_\_\_\_\_\_$$

- **Step 3:** Determine if there is a local extremum at each critical number using a graphing utility.

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# Locating Critical Numbers: Example c

For the function $f(x) = \frac{4x}{1+x^2}$:

- **Step 1:** Find the derivative $f'(x)$.

$$f'(x) = \_\_\_\_\_\_\_\_$$

- **Step 2:** Set $f'(x) = 0$ and solve for $x$ to find the critical numbers.

$$f'(x) = 0 \quad \Rightarrow \quad x = \_\_\_\_\_\_\_\_$$

- **Step 3:** Determine if there is a local extremum at each critical number using a graphing utility.

$$\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

# Finding Critical Numbers

Find all critical numbers for the function:

$$f(x) = x^3 - \frac{1}{2}x^2 - 2x + 1$$

**Step 1:** Find the derivative $f'(x)$.

$$f'(x) = \_\_\_\_\_\_\_\_\_\_\_\_$$

**Step 2:** Set the derivative equal to zero and solve for $x$.

$$f'(x) = 0 \quad \Rightarrow \quad x = \_\_\_\_\_\_\_\_$$

**Step 3:** List the critical numbers:

Critical numbers: ___________

# Location of Absolute Extrema

**Let $I = [a, b]$**

If $f$ is a continuous function on a closed interval $I$, then:

- The **absolute maximum**
- The **absolute minimum**

must happen either at the endpoints of $I$ or at critical points inside $I$.

# Problem-Solving Strategy: Finding Absolute Extrema

Consider a continuous function $f$ defined over the closed interval $[a, b]$.

1. **Step 1: Evaluate** the function $f$ at the endpoints of the interval. Calculate:

$$f(a) \quad \text{and} \quad f(b)$$

2. **Step 2: Find** all critical numbers of the function $f$ within the open interval $(a, b)$. A critical number is where:

$$f'(x) = 0 \quad \text{or} \quad f'(x) \text{ is undefined}$$

   Then, evaluate the function at each critical number found.

3. **Step 3: Compare** all values obtained from Steps 1 and 2:
   - The largest value is the **absolute maximum**.
   - The smallest value is the **absolute minimum**.

# Locating Absolute Extrema for Function a

**Function a:** $f(x) = -x^2 + 3x - 2$ over $[1, 3]$

1. **Step 1:** Find the derivative $f'(x)$.

$$f'(x) = \_\_\_\_\_\_\_\_\_\_$$

2. **Step 2:** Solve $f'(x) = 0$ to find the critical numbers.

$$f'(x) = 0 \quad \Rightarrow \quad x = \_\_\_\_\_\_$$

3. **Step 3:** Evaluate $f(x)$ at the critical points and endpoints $x = 1$ and $x = 3$.

$$f(1) = \_\_\_\_\_\_ \quad f(3) = \_\_\_\_\_\_$$

4. **Step 4:** Compare the values and determine the absolute maximum and minimum.

Max: ________ Min: ________

# Locating Absolute Extrema for Function b

**Function b:** $f(x) = x^2 - 3x^{2/3}$ over $[0, 2]$

1. **Step 1:** Find the derivative $f'(x)$.

$$f'(x) = \_\_\_\_\_\_\_\_\_\_$$

2. **Step 2:** Solve $f'(x) = 0$ to find the critical numbers.

$$f'(x) = 0 \quad \Rightarrow \quad x = \_\_\_\_\_\_$$

3. **Step 3:** Evaluate $f(x)$ at the critical points and endpoints $x = 0$ and $x = 2$.

$$f(0) = \_\_\_\_\_\_ \quad f(2) = \_\_\_\_\_\_$$

4. **Step 4:** Compare the values and determine the absolute maximum and minimum.

Max: ________ Min: ________

# Locating Absolute Extrema

Find the absolute maximum and absolute minimum of $f(x) = x^2 - 4x + 3$ over the interval $[1, 4]$.

1. **Step 1:** Find the derivative $f'(x)$.

$$f'(x) = \_\_\_\_\_\_\_\_\_\_$$

2. **Step 2:** Solve $f'(x) = 0$ to find the critical numbers.

$$f'(x) = 0 \quad \Rightarrow \quad x = \_\_\_\_\_\_\_$$

3. **Step 3:** Evaluate $f(x)$ at the critical points and the endpoints $x = 1$ and $x = 4$.

$$f(1) = \_\_\_\_\_\_\_ \quad f(4) = \_\_\_\_\_\_\_$$

4. **Step 4:** Compare the values to determine the absolute maximum and minimum.

Max: _________ Min: ________

# Formula for the Maximum or Minimum of a Quadratic

**Problem:** In precalculus, you learned a formula for the position of the maximum or minimum of a quadratic equation $y = ax^2 + bx + c$, which was:

$$m = -\frac{b}{2a}$$

Prove this formula using calculus.

1. **Step 1:** Start with the given quadratic function:

$$y = ax^2 + bx + c$$

2. **Step 2:** Find the derivative of $y$, $y'(x)$.

$$y'(x) = \_\_\_\_\_\_\_$$

3. **Step 3:** Set $y'(x) = 0$ and solve for $x$ to find the critical point.

$$0 = \_\_\_\_\_\_\_ \quad \Rightarrow \quad x = \_\_\_\_\_\_\_$$

4. **Step 4:** Conclude that the critical point $x = \frac{-b}{2a}$ gives the position of the maximum or minimum.

# Key Concepts

- A function may have both an absolute maximum and an absolute minimum, have just one absolute extremum, or have no absolute maximum or absolute minimum.
- If a function has a local extremum, the point at which it occurs must be a critical number. However, a function need not have a local extremum at a critical number.
- A continuous function over a closed, bounded interval has an absolute maximum and an absolute minimum. Each extremum occurs at a critical number or an endpoint.

# Meam Value Thorem

Clotilde Djuikem

# Learning Objectives

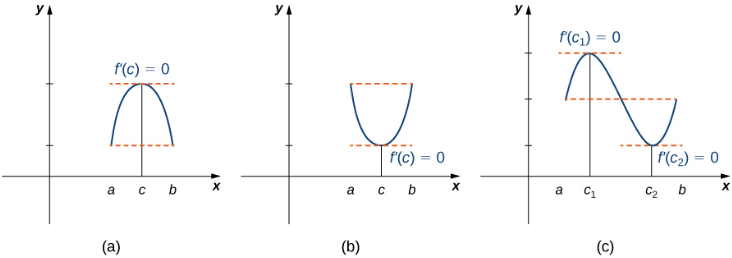
- Explain the meaning of Rolle's theorem.
- Describe the significance of the Mean Value Theorem.
- State three important consequences of the Mean Value Theorem.

## Definition

If the outputs of a differentiable function $f$ are equal at the endpoints of an interval, there must be an interior point $c$ where $f'(c) = 0$.

## Visual Illustration

See the figure with parabolas and sine wave, illustrating different cases of Rolle's theorem.



# Formal Statement of Rolle's Theorem

**Statement**

Let $f$ be a

- Continuous function over the closed interval $[a, b]$ and
- Differentiable over the open interval $(a, b)$ such that $f(a) = f(b)$.

Then, there exists at **least one** $c \in (a, b)$ such that $f'(c) = 0$.

# Using Rolle's Theorem

## Example

For the function $f(x) = x^2 + 2x$ over $[-2, 0]$, verify the criteria of Rolle's theorem and find the value $c$ where $f'(c) = 0$.

## Solution

Since $f(x)$ is continuous and differentiable on the interval, and $f(-2) = f(0) = 0$, there exists a point $c = -1$ where $f'(c) = 0$.

# Example : Using Rolle's Theorem

**Problem:** Let $f(x) = x^2 - 4x + 4$ on the interval $[0, 4]$. Verify that $f$ satisfies the conditions of Rolle's theorem and find the point $c$ such that $f'(c) = 0$.

**Solution:**

1. **Check continuity and differentiability:**
   $f(x) = x^2 - 4x + 4$ is a polynomial, so it is continuous and differentiable on $[0, 4]$.
2. **Verify endpoint equality:**
   $f(0) = 4$ and $f(4) = 4$. Since $f(0) = f(4)$, the conditions of Rolle's theorem are satisfied.
3. **Differentiate $f(x)$:**
   $f'(x) = 2x - 4$.
4. **Set $f'(c) = 0$ and solve for $c$:**
   $2c - 4 = 0 \Rightarrow c = 2$.
5. **Conclusion:**
   There exists a point $c = 2$ such that $f'(c) = 0$, which satisfies Rolle's theorem.

# Example : Verifying Conditions for Rolle's Theorem

**Problem:** Let $f(x) = x^3 - 3x + 2$ over $[-2, 2]$. Determine if Rolle's theorem applies and find the value(s) of $c$ if applicable.

**Solution:**

1. **Continuity and Differentiability:**
   $f(x) = x^3 - 3x + 2$ is a polynomial, so it is continuous and differentiable on $[-2, 2]$.
2. **Check endpoint equality:**
   $f(-2) = 0$ and $f(2) = 0$, so $f(-2) = f(2)$.
3. **Differentiate $f(x)$:** $f'(x) = 3x^2 - 3$.
4. **Solve $f'(c) = 0$:**

$$3c^2 - 3 = 0 \Rightarrow c^2 = 1 \Rightarrow c = \pm 1.$$

5. **Conclusion:**
   The points $c = 1$ and $c = -1$ satisfy $f'(c) = 0$, as required by Rolle's theorem.

# Exercise

**Problem:** Let $f(x) = x^2 - 5x + 6$ on $[1, 3]$. Use Rolle's theorem to verify the conditions and find the value(s) of $c$ if possible.

**Solution:**

1. **Continuity and Differentiability:**

   ---

2. **Verify endpoint equality:**

   ---

3. **Differentiate $f(x)$:**

   ---

4. **Solve $f'(c) = 0$:**

   ---

5. **Conclusion:**

   ---

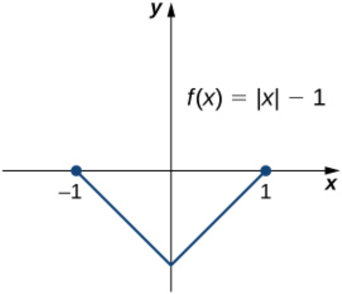
# Rolle's Theorem: Importance of Differentiability

**Example:** Let $f(x) = |x| - 1$ on $[-1, 1]$.

- $f(x)$ is continuous on $[-1, 1]$ and $f(-1) = f(1) = 0$.
- However, $f$ is not differentiable at $x = 0$.

## Conclusion

Since $f$ is not differentiable at $x = 0$, Rolle's theorem does not apply. There is no $c \in (-1, 1)$ such that $f'(c) = 0$.




No c such that f′(c) = 0

# Using Rolle's Theorem

**Problem:** For each of the following functions, verify that the function satisfies the criteria of Rolle's theorem and find all values $c$ in the given interval where $f'(c) = 0$.

**a.** $f(x) = x^2 + 2x$ over $[-2, 0]$

**Solution Steps:**

1. **Check continuity and differentiability:**

   ________________________________________

2. **Verify endpoint equality:**
   $f(-2) =$ ________, $f(0) =$ ________
   Since $f(-2) = f(0)$, the conditions are satisfied.

3. **Differentiate $f(x)$:**
   $f'(x) =$ __________________________

4. **Set $f'(c) = 0$ and solve for $c$:**
   $f'(c) = 0 \Rightarrow c =$ ________________

# Using Rolle's Theorem

**b.** $f(x) = x^3 - 4x$ over $[-2, 2]$

**Solution Steps:**

1. **Check continuity and differentiability:**

   ________________________________

2. **Verify endpoint equality:**

   $f(-2) =$ ________, $f(2) =$ ________

   Since $f(-2) = f(2)$, the conditions are satisfied.

3. **Differentiate $f(x)$:**

   $f'(x) =$ ____________________

4. **Set $f'(c) = 0$ and solve for $c$:**

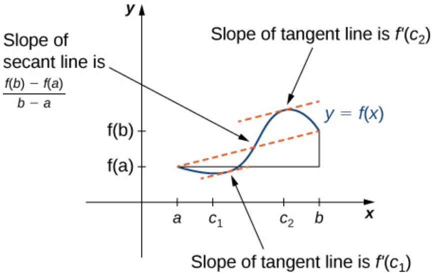
   $f'(c) = 0 \Rightarrow c =$ ________________

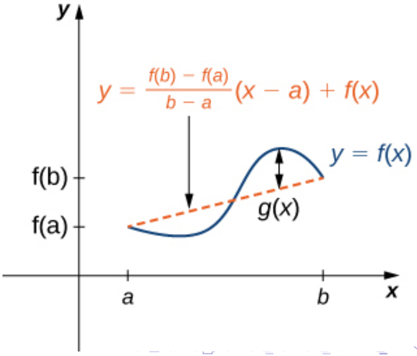
# The Mean Value Theorem (MVT)

## Statement

Let $f$ be continuous over the closed interval $[a, b]$ and differentiable over the open interval $(a, b)$. Then there exists a point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$



# Applying the Mean Value Theorem

**Example**

For the function $f(x) = \sqrt{x}$ over $[0, 9]$, find the point $c$ such that $f'(c)$ equals the slope of the secant line between $(0, f(0))$ and $(9, f(9))$.

**Solution**

The slope of the secant line is $\frac{3}{9} = \frac{1}{3}$. We find $c = \frac{9}{4}$ such that $f'(c) = \frac{1}{3}$.

# Example 2: Applying the Mean Value Theorem

**Problem:** Let $f(x) = \sqrt{x}$ on the interval $[1, 4]$. Use the Mean Value Theorem to find the value $c$ such that $f'(c)$ equals the slope of the secant line between $(1, f(1))$ and $(4, f(4))$. **Solution:**

1. **Verify continuity and differentiability:** $f(x) = \sqrt{x}$ is continuous on $[1, 4]$ and differentiable on $(1, 4)$.
2. **Calculate the secant slope:**
$$\text{slope} = \frac{f(4) - f(1)}{4 - 1} = \frac{2 - 1}{3} = \frac{1}{3}.$$
3. **Differentiate $f(x)$:** $f'(x) = \frac{1}{2\sqrt{x}}$.
4. **Set $f'(c) = \frac{1}{3}$ and solve for $c$:**
$$\frac{1}{2\sqrt{c}} = \frac{1}{3} \Rightarrow \sqrt{c} = \frac{3}{2} \Rightarrow c = \frac{9}{4}.$$
5. **Conclusion:** At $c = \frac{9}{4}$, the tangent slope equals the secant slope, satisfying the Mean Value Theorem.

# Exercise

**Problem:** Let $f(x) = x^2 + x - 12$ on $[2, 6]$. Use the Mean Value Theorem to find $c$ such that $f'(c)$ equals the secant slope.

**Solution:**

1. **Continuity and Differentiability:**

   ______________________________

2. **Calculate the secant slope:**

   ______________________________

3. **Differentiate $f(x)$:**

   ______________________________

4. **Set $f'(c)$ equal to the secant slope and solve for $c$:**

   ______________________________

5. **Conclusion:**

   ______________________________

# Mean Value Theorem and Inequalities

## Problem

Use the Mean Value Theorem to show that if $x > 0$, then $\sin x \leq x$.

**Problem**

Use the Mean Value Theorem to show that if $x > 0$, then $\sin x \leq x$.

Solution **Step 1: Define the function and interval.**
$f(x) = \sin x - x$. To show that $f(x) \leq 0$, we consider the interval $[0, x]$.
**Step 2: Check continuity and differentiability.** The function $f$ is the difference of a trigonometric function and a polynomial. Thus, $f$ is continuous on $[0, x]$ and differentiable on $(0, x)$.
**Step 3: Calculate the derivative of $f(x)$.** We find that $f'(x) = \cos x - 1$.
**Step 4: Apply the Mean Value Theorem.** By the Mean Value Theorem, there exists a point $c \in (0, x)$ such that

$$f'(c) = \frac{f(x) - f(0)}{x - 0} = \frac{f(x)}{x}.$$

**Step 5: Analyze $f'(c)$ to conclude.**
Since $f'(c) = \cos c - 1 \leq 0$ (because $\cos c \leq 1$ for all $c$), and $x > 0$, we conclude that $f(x) \leq 0$. Therefore, $\sin x \leq x$.

# Problem: Ball Dropped from a Height

**Problem:** A ball is dropped from a height of 200 ft. Its position at time $t$ is $s(t) = -16t^2 + 200$. Find the time $t$ when the instantaneous velocity equals the average velocity.

**Hint**

1. Find the time it takes for the ball to hit the ground.
2. Calculate the average velocity.

# Problem: Ball Dropped from a Height

**Problem:** A ball is dropped from a height of 200 ft. Its position at time $t$ is $s(t) = -16t^2 + 200$. Find the time $t$ when the instantaneous velocity equals the average velocity.

**Hint**

1. Find the time it takes for the ball to hit the ground.
2. Calculate the average velocity.

Solution **Step 1: Time to hit the ground**

$$s(t) = 0 \Rightarrow -16t^2 + 200 = 0 \Rightarrow t = \frac{5\sqrt{2}}{2}.$$

**Step 2: Average velocity** $v_{\text{avg}} = \frac{s(t)-s(0)}{t} = -40\sqrt{2}$ ft/sec.
**Step 3: Instantaneous velocity**
$s'(t) = -32t$. Set $s'(t) = -40\sqrt{2}$:

$$-32t = -40\sqrt{2} \Rightarrow t = \frac{5\sqrt{2}}{2} \text{seconds.}$$

# Corollaries of the Mean Value Theorem

**Corollary 1: Functions with a Derivative of Zero**

Let $f$ be differentiable over an interval $I$. If $f'(x) = 0$ for all $x \in I$, then $f(x)$ is constant for all $x \in I$.

**Corollary 2: Constant Difference Theorem**

If $f$ and $g$ are differentiable over an interval $I$ and $f'(x) = g'(x)$ for all $x \in I$, then

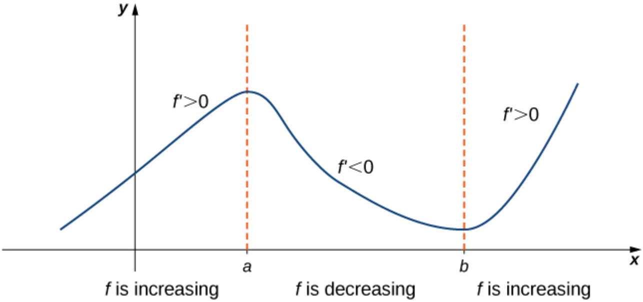
$$f(x) = g(x) + C$$

for some constant $C$.

# Corollaries of the Mean Value Theorem

## Key Corollaries

- If $f'(x) = 0$ over an interval $I$, then $f$ is constant over $I$.
- If $f'(x) > 0$ over $I$, then $f$ is increasing over $I$.
- If $f'(x) < 0$ over $I$, then $f$ is decreasing over $I$.



# Key Concepts

## Rolle's Theorem

If $f$ is continuous over $[a, b]$ and differentiable over $(a, b)$, and $f(a) = f(b) = 0$, then there exists a point $c \in (a, b)$ such that $f'(c) = 0$.

## Mean Value Theorem (MVT)

If $f$ is continuous over $[a, b]$ and differentiable over $(a, b)$, then there exists a point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

## Constant Function Property

If $f'(x) = 0$ over an interval $I$, then $f$ is constant over $I$.

## Equality of Derivatives Implies Constant Difference

If two differentiable functions $f$ and $g$ satisfy $f'(x) = g'(x)$ over an interval $I$, then $f(x) = g(x) + C$ for some constant $C$.

# Key Concepts

**Monotonicity**

If $f'(x) > 0$ over an interval $I$, then $f$ is increasing over $I$. If $f'(x) < 0$ over $I$, then $f$ is decreasing over $I$.

# Derivatives and the Shape of a Graph

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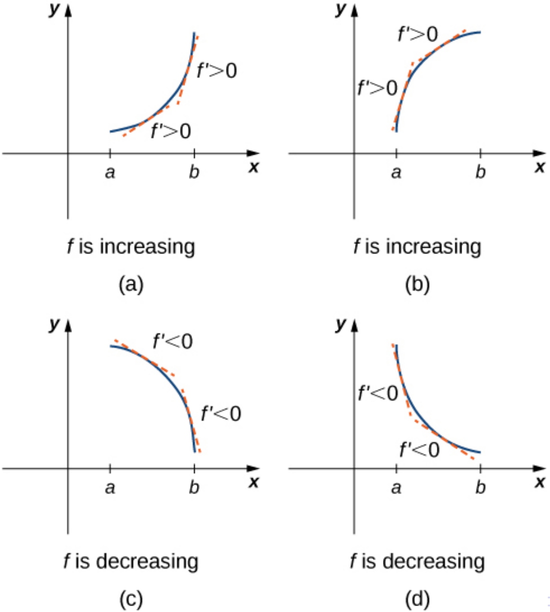
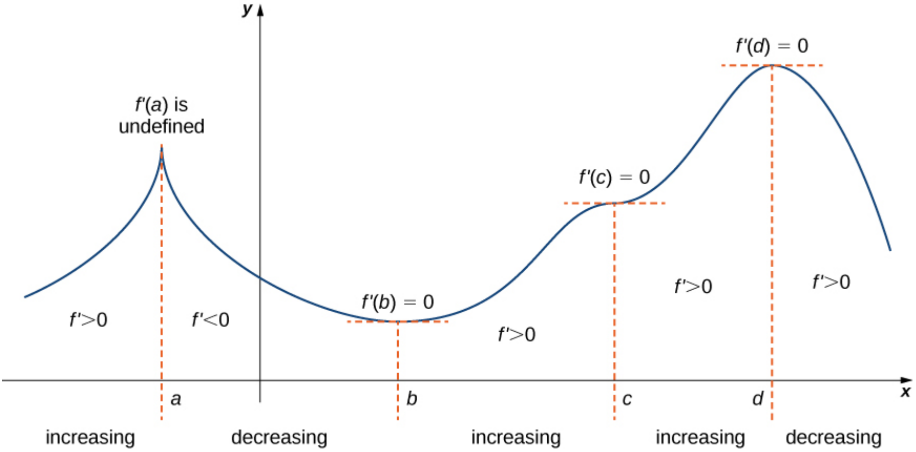
# Learning Objectives

- Explain how the sign of the first derivative affects the shape of a function's graph.
- State the first derivative test for identifying critical numbers.
- Use concavity and inflection points to explain how the sign of the second derivative affects the shape of a function's graph.
- Explain the concavity test for a function over an open interval.
- Describe the relationship between a function and its first and second derivatives.
- State the second derivative test for identifying local extrema.

# Graph of function and sign of derivative



# Graph


y
x
f′(a) is undefined
f′(b) = 0
f′(c) = 0
f′(d) = 0
f′>0
f′<0
f′>0
f′>0
f′>0
a
b
c
d
increasing
decreasing
increasing
increasing
decreasing

# First Derivative Test

## First Derivative Test

Suppose that $f$ is a continuous function over an interval $I$ containing a critical number $c$. If $f$ is differentiable over $I$, except possibly at $c$, then $f(c)$ satisfies one of the following conditions:

1. If $f'$ changes sign from positive to negative at $c$, then $f(c)$ is a local maximum.
2. If $f'$ changes sign from negative to positive at $c$, then $f(c)$ is a local minimum.
3. If $f'$ has the same sign on both sides of $c$, then $f(c)$ is not a local extremum.

# Problem-Solving Strategy: First Derivative Test

For a continuous function $f$ over interval $I$:

1. **Identify Critical Numbers:** Find points where $f'(x) = 0$ or $f'(x)$ is undefined.
2. **Determine $f'$ Sign in Each Subinterval:**
    - Select a test point in each subinterval.
    - Check if $f'(x)$ is positive (increasing) or negative (decreasing).
3. **Conclude Local Behavior at Each Critical Number:**
    - $f'$ changes $+ \to -$: local max.
    - $f'$ changes $- \to +$: local min.
    - $f'$ does not change: no extremum.

# Finding Local Extrema Using the First Derivative Test

**Problem:** Use the first derivative test to find the location of all local extrema for $f(x) = x^3 - 3x^2 - 9x - 1$. Confirm your results using a graphing utility. **Solution Steps:**

1. **Find the derivative:** $f'(x) = 3x^2 - 6x - 9$.
2. **Set $f'(x) = 0$ and solve for $x$:**

$$3x^2 - 6x - 9 = 0 \Rightarrow x^2 - 2x - 3 = 0 \Rightarrow (x-3)(x+1) = 0$$

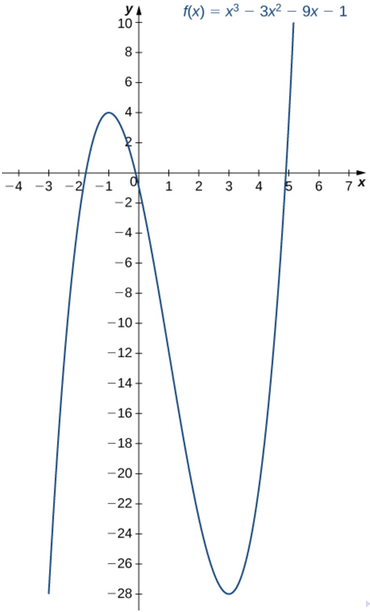
   Critical points: $x = 3$ and $x = -1$.

3. **Test the sign of $f'$ around each critical point:**

| $x$ Interval | $(-\infty, -1)$ | $(-1, 3)$ | $(3, \infty)$ |
|---|---|---|---|
| Test Point $x$ | $-2$ | $0$ | $4$ |
| $f'(x)$ | $+$ (positive) | $-$ (negative) | $+$ (positive) |
| $f(x)$ Behavior | Increasing | Decreasing | Increasing |

4. **Determine local extrema:**
   - $f'$ changes from positive to negative at $x = -1$: local maximum.
   - $f'$ changes from negative to positive at $x = 3$: local minimum.

# Confirm results with a graphing utility.



# Finding Local Extrema Using the First Derivative Test

**Problem:** Use the first derivative test to locate all local extrema for $f(x) = -x^3 + \frac{3}{2}x^2 + 18x$.

**Solution Steps:**

1. **Find the derivative:** $f'(x) = -3x^2 + 3x + 18$
2. **Set $f'(x) = 0$ and solve for $x$:**

$$-3x^2 + 3x + 18 = 0 \Rightarrow x^2 - x - 6 = 0 \Rightarrow (x-3)(x+2) = 0$$

   Critical points: $x = 3$ and $x = -2$.

3. **Test the sign of $f'$ around each critical point:**

| $x$ Interval | $(-\infty, -2)$ | $(-2, 3)$ | $(3, \infty)$ |
|---|---|---|---|
| Test Point $x$ | $-3$ | $0$ | $4$ |
| $f'(x)$ | $+$ (positive) | $-$ (negative) | $+$ (positive) |
| $f(x)$ Behavior | Increasing | Decreasing | Increasing |

4. **Determine local extrema:**
   - $f'$ changes from positive to negative at $x = -2$: local maximum.
   - $f'$ changes from negative to positive at $x = 3$: local minimum.

# Using the First Derivative Test

**Problem:** Use the first derivative test to find the location of all local extrema for $f(x) = 5x^{1/3} - x^{5/3}$. Confirm your results using a graphing utility. **Solution Steps:**

1. **Find the derivative:**

$$f'(x) = \frac{5}{3}x^{-2/3} - \frac{5}{3}x^{2/3} = \frac{5}{3}\left(x^{-2/3} - x^{2/3}\right) = \frac{5}{3} \cdot \frac{1-x}{x^{2/3}}$$

2. **Set $f'(x) = 0$ and solve for $x$:** $\frac{5}{3} \cdot \frac{1-x}{x^{2/3}} = 0 \Rightarrow 1 - x = 0$ Critical point: $x = 1$. Note that $f'(x)$ is undefined at $x = 0$, so $x = 0$ is also a critical point.

3. **Test the sign of $f'$ around each critical point:**

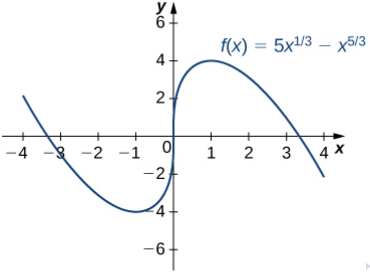
| $x$ Interval | $(-\infty, 0)$ | $(0, 1)$ | $(1, \infty)$ |
|---|---|---|---|
| Test Point $x$ | $-1$ | 0.5 | 2 |
| $f'(x)$ | $-$ (negative) | $+$ (positive) | $-$ (negative) |
| $f(x)$ Behavior | Decreasing | Increasing | Decreasing |

4. **Determine local extrema:**
   - $f'$ changes from negative to positive at $x = 0$: local minimum.
   - $f'$ changes from positive to negative at $x = 1$: local maximum.

## Graph

**Confirm results with a graphing utility:** Plot $f(x)$ to verify the local minimum at $x = 0$ and local maximum at $x = 1$.

Since $f$ is decreasing over the interval $(-\infty, -1)$ and increasing over $(-1, 0)$, $f$ has a local minimum at $x = -1$. Since $f$ is increasing over both $(-1, 0)$ and $(0, 1)$, $f$ does not have a local extremum at $x = 0$. Since $f$ is increasing over $(0, 1)$ and decreasing over $(1, \infty)$, $f$ has a local maximum at $x = 1$. These analytical results are confirmed by the following graph.



# Using the First Derivative Test

**Problem:** Use the first derivative test to find all local extrema for $f(x) = \sqrt[3]{x} - 1$.

**Hint:** The only critical number of $f$ is $x = 1$.

# Using the First Derivative Test

**Problem:** Use the first derivative test to find all local extrema for $f(x) = \sqrt[3]{x} - 1$.

**Hint:** The only critical number of $f$ is $x = 1$. **Solution Steps:**

1. **Find the derivative:**

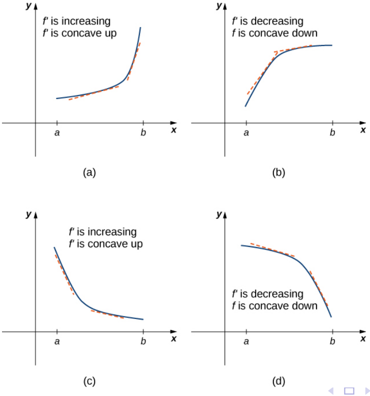
$$f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3\sqrt[3]{x^2}}$$

2. **Identify critical points:** $f'(x)$ is undefined at $x = 0$ and equal to zero at $x = 1$. Thus, the critical number is $x = 1$.
3. **Test the sign of $f'$ around $x = 1$:**
   - For $x = 0.5$ (left of $x = 1$): $f'(0.5) = \frac{1}{3 \cdot \sqrt[3]{0.5^2}} > 0$ (positive).
   - For $x = 2$ (right of $x = 1$): $f'(2) = \frac{1}{3 \cdot \sqrt[3]{2^2}} > 0$ (positive).
4. **Conclusion:** Since $f'(x)$ is positive on both sides of $x = 1$, there is no local extremum at $x = 1$.

## Concavity of a Function

Let $f$ be a function that is differentiable over an open interval $I$.

- If $f'$ is increasing over $I$, we say $f$ is **concave up** over $I$.
- If $f'$ is decreasing over $I$, we say $f$ is **concave down** over $I$.



# Test for Concavity with Examples

## Concavity Test

Let $f$ be a function that is twice differentiable over an interval $I$.

1. If $f''(x) > 0$ for all $x \in I$, then $f$ is **concave up** over $I$.
2. If $f''(x) < 0$ for all $x \in I$, then $f$ is **concave down** over $I$.

**Examples:**

- **Example 1:** $f(x) = x^2$
  - $f'(x) = 2x$, $f''(x) = 2$
  - Since $f''(x) = 2 > 0$ for all $x$, $f(x) = x^2$ is **concave up** everywhere.
- **Example 2:** $f(x) = -x^2$
  - $f'(x) = -2x$, $f''(x) = -2$
  - Since $f''(x) = -2 < 0$ for all $x$, $f(x) = -x^2$ is **concave down** everywhere.

# Definition of an Inflection Point

## Inflection Point

If $f$ is continuous at $a$ and $f$ changes concavity at $a$, then the point $(a, f(a))$ is an **inflection point** of $f$.

**Example 1:** Determine the inflection points of $f(x) = x^3 - 3x^2 + 4$.

- Find $f''(x)$ and set it equal to zero to find potential inflection points.
- Verify if $f$ changes concavity at these points.

**Solution:**

- $f'(x) = 3x^2 - 6x$
- $f''(x) = 6x - 6$
- Set $f''(x) = 0 \Rightarrow x = 1$
- Check concavity around $x = 1$:
    - $f''(x) > 0$ for $x > 1$ (concave up)
    - $f''(x) < 0$ for $x < 1$ (concave down)
- Conclusion: $(1, f(1)) = (1, 2)$ is an inflection point.

# Second Derivative Test

## Second Derivative Test

Suppose $f'(c) = 0$ and $f''(x)$ is continuous over an interval containing $c$.

1. If $f''(c) > 0$, then $f$ has a **local minimum** at $c$.
2. If $f''(c) < 0$, then $f$ has a **local maximum** at $c$.
3. If $f''(c) = 0$, then the test is **inconclusive**.

# Using the Second Derivative Test

**Problem:** Use the second derivative to find the location of all local extrema for $f(x) = x^5 - 5x^3$.

# Using the Second Derivative Test

**Problem:** Use the second derivative to find the location of all local extrema for $f(x) = x^5 - 5x^3$. **Solution Steps:**

1. **Find the first derivative:**

$$f'(x) = 5x^4 - 15x^2 = 5x^2(x^2 - 3)$$

   Set $f'(x) = 0$: $5x^2(x^2 - 3) = 0 \Rightarrow x = 0 \quad \text{and} \quad x = \pm\sqrt{3}$ Critical points: $x = 0$, $x = \sqrt{3}$, and $x = -\sqrt{3}$.

2. **Find the second derivative:**

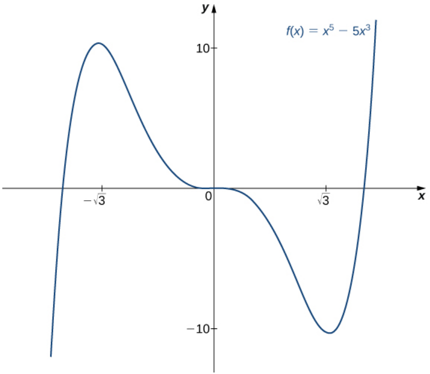
$$f''(x) = 20x^3 - 30x = 10x(2x^2 - 3)$$

3. **Evaluate $f''(x)$ at each critical point:**
   - $f''(0) = 10 \cdot 0 \cdot (2 \cdot 0^2 - 3) = 0$ (inconclusive).
   - $f''(\sqrt{3}) = 10\sqrt{3}(2 \cdot 3 - 3) = 30\sqrt{3} > 0$ (local minimum).
   - $f''(-\sqrt{3}) = 10(-\sqrt{3})(2 \cdot 3 - 3) = -30\sqrt{3} < 0$ (local maximum).

4. **Conclusion:**
   - $f(x)$ has a local minimum at $x = \sqrt{3}$.
   - $f(x)$ has a local maximum at $x = -\sqrt{3}$.
   - The second derivative test is inconclusive at $x = 0$.

# The second derivative is inconcluse

- **Conclusion:** Since $f'$ is negative on both intervals around $x = 0$, $f$ is decreasing across $x = 0$. Therefore, $f$ does not have a local extremum at $x = 0$.
- The graph confirms these results.



# Example 1: Polynomial Function

**Problem:** Use the second derivative test to find the location of all local extrema for $f(x) = x^4 - 4x^2$.

# Example 1: Polynomial Function

**Problem:** Use the second derivative test to find the location of all local extrema for $f(x) = x^4 - 4x^2$.

**Solution Steps:**

1. **Find the First Derivative:**

$$f'(x) = 4x(x^2 - 2) \Rightarrow x = 0, \pm\sqrt{2}$$

   Critical points: $x = 0$ and $x = \pm\sqrt{2}$.

2. **Find the Second Derivative:**

$$f''(x) = 12x^2 - 8$$

3. **Evaluate $f''(x)$ at Each Critical Point:**
   - $f''(0) = -8$: local maximum at $x = 0$.
   - $f''(\pm\sqrt{2}) = 16$: local minima at $x = \pm\sqrt{2}$.
4. **Conclusion:**
   - Local maximum at $x = 0$.
   - Local minima at $x = \pm\sqrt{2}$.

# Example 2: Exponential Function

**Problem:** Use the second derivative test to find the location of all local extrema for $f(x) = e^{-x^2}$.

# Example 2: Exponential Function

**Problem:** Use the second derivative test to find the location of all local extrema for $f(x) = e^{-x^2}$.

**Solution Steps:**

1. **Find the First Derivative:**

$$f'(x) = -2xe^{-x^2}$$

   Set $f'(x) = 0$: Critical point is $x = 0$.

2. **Find the Second Derivative:**

$$f''(x) = (4x^2 - 2)e^{-x^2}$$

3. **Evaluate $f''(x)$ at the Critical Point $x = 0$:**

$$f''(0) = -2 \Rightarrow \text{local maximum at } x = 0$$

4. **Conclusion:**
   - Local maximum at $x = 0$.

# Example 3: Trigonometric Function

**Problem:** Use the second derivative test to find the location of all local extrema for $f(x) = \sin(x) + \cos(x)$ over the interval $[0, 2\pi]$.

# Example 3: Trigonometric Function

**Problem:** Use the second derivative test to find the location of all local extrema for $f(x) = \sin(x) + \cos(x)$ over the interval $[0, 2\pi]$.

**Solution Steps:**

1. **Find the First Derivative:**

$$f'(x) = \cos(x) - \sin(x)$$

   Set $f'(x) = 0$: Solving $\cos(x) = \sin(x)$ gives critical points $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$.

2. **Find the Second Derivative:**

$$f''(x) = -\sin(x) - \cos(x)$$

3. **Evaluate $f''(x)$ at Each Critical Point:**
   - $f''\left(\frac{\pi}{4}\right) = -\sqrt{2}$: local maximum at $x = \frac{\pi}{4}$.
   - $f''\left(\frac{5\pi}{4}\right) = \sqrt{2}$: local minimum at $x = \frac{5\pi}{4}$.

4. **Conclusion:**
   - Local maximum at $x = \frac{\pi}{4}$.
   - Local minimum at $x = \frac{5\pi}{4}$.

# Key Concepts

## Critical Points and Sign of $f'$

- If $c$ is a critical number of $f$ and $f'(x) > 0$ for $x < c$ and $f'(x) < 0$ for $x > c$, then $f$ has a local maximum at $c$.
- If $c$ is a critical number of $f$ and $f'(x) < 0$ for $x < c$ and $f'(x) > 0$ for $x > c$, then $f$ has a local minimum at $c$.

## Concavity

- If $f''(x) > 0$ over an interval $I$, then $f$ is concave up over $I$.
- If $f''(x) < 0$ over an interval $I$, then $f$ is concave down over $I$.

## Second Derivative Test

- If $f'(c) = 0$ and $f''(c) > 0$, then $f$ has a local minimum at $c$.
- If $f'(c) = 0$ and $f''(c) < 0$, then $f$ has a local maximum at $c$.
- If $f'(c) = 0$ and $f''(c) = 0$, use the First Derivative Test or evaluate $f'(x)$ at points around $c$ to determine if $f$ has a local extremum at $c$.

# Applied Optimization Problems

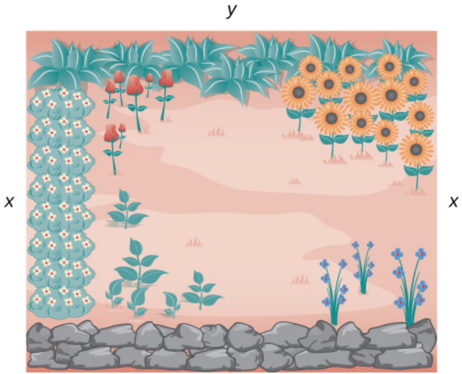
Clotilde Djuikem

# Learning Objective

- Set up and solve optimization problems in several applied fields.

# Maximizing the Area of a Garden

A rectangular garden is to be constructed using a rock wall as one side of the garden and wire fencing for the other three sides . Given 100 ft of wire fencing, determine the dimensions that would create a garden of maximum area. What is the maximum area?



# Problem Setup

**Objective:** Maximize the area of a rectangular garden.

Let:

- $x$: Length of the side perpendicular to the rock wall.
- $y$: Length of the side parallel to the rock wall.

**Given:**

- Total fencing available: 100 ft.
- Area of the garden: $A = x \cdot y$.
- Constraint: $2x + y = 100$.

# Step 1: Express Area in Terms of One Variable

From the constraint equation:

$$2x + y = 100 \quad \Longrightarrow \quad y = 100 - 2x.$$

Substitute $y = 100 - 2x$ into $A = x \cdot y$:

$$A(x) = x \cdot (100 - 2x) = 100x - 2x^2.$$

Thus, the area is given by:

$$A(x) = 100x - 2x^2.$$

# Step 2: Determine the Domain

To construct a rectangular garden:

- Both $x$ and $y$ must be positive:

$$x > 0 \quad \text{and} \quad y = 100 - 2x > 0.$$

- This implies $x < 50$.

Therefore, the domain for $x$ is:

$$0 < x < 50.$$

To use the Extreme Value Theorem, we extend this to the closed interval:

$$[0, 50].$$

# Step 3: Find the Critical Number

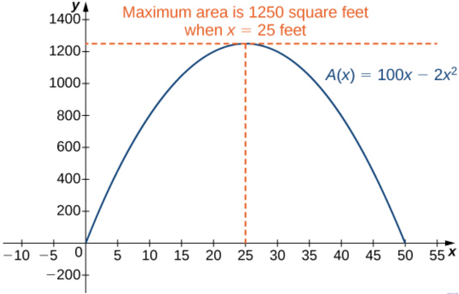
Differentiate $A(x) = 100x - 2x^2$:

$$A'(x) = 100 - 4x.$$

Set $A'(x) = 0$ to find the critical number:

$$100 - 4x = 0 \quad \Longrightarrow \quad x = 25.$$

The critical number is $x = 25$.



# Step 4: Evaluate the Area

Evaluate $A(x)$ at the endpoints and critical number:

- At $x = 0$: $A(0) = 100(0) - 2(0)^2 = 0$.
- At $x = 50$: $A(50) = 100(50) - 2(50)^2 = 0$.
- At $x = 25$:

$$A(25) = 100(25) - 2(25)^2 = 2500 - 1250 = 1250\,\text{ft}^2.$$

**Maximum Area:** $1250\,\text{ft}^2$ occurs at $x = 25$.

# Solution Summary

**Optimal Dimensions:**

- $x = 25\,\text{ft}$ (perpendicular to the rock wall).
- $y = 100 - 2(25) = 50\,\text{ft}$ (parallel to the rock wall).

**Maximum Area:**

$$A = x \cdot y = 25 \cdot 50 = 1250\,\text{ft}^2.$$

**Conclusion:** To maximize the area, construct a garden with dimensions $25\,\text{ft} \times 50\,\text{ft}$.

# Problem-Solving Strategy: Optimization Problems

**Step-by-Step Approach:**

1. **Introduce all variables:** Define variables and, if applicable, draw and label a diagram.
2. **Identify the target quantity:** Determine what needs to be maximized or minimized and specify the range of possible values for other variables.
3. **Write the formula:** Express the quantity to optimize in terms of the variables.
4. **Relate variables:** Use additional equations or constraints to rewrite the formula as a function of one variable.
5. **Determine the domain:** Identify valid values for the variable(s) based on the physical context of the problem.
6. **Find the optimal value:** Differentiate the function, locate critical numbers, and justify the maximum or minimum using appropriate methods.
7. **State the final answer:** Provide a clear sentence with units, ensuring the solution satisfies the problem's constraints.

# Maximizing the Volume of a Box

An open-top box is to be made from a 24 in. by 36 in. piece of cardboard by removing a square from each corner of the box and folding up the flaps on each side. What size square should be cut out of each corner to get a box with the maximum volume?

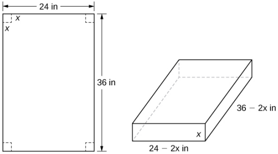
# Problem Setup and Volume Function

**Objective:** Maximize the volume of an open-top box formed by cutting squares from a rectangular sheet.

**Setup:**

- Dimensions of cardboard: 36 in by 24 in.
- $x$: Side length of the square cut from each corner (in inches).
- Box dimensions after folding:
    - Height: $x$,
    - Length: $36 - 2x$,
    - Width: $24 - 2x$.
- Volume formula:

$$V(x) = (36 - 2x)(24 - 2x)x = 4x^3 - 120x^2 + 864x.$$



# Domain and Critical Numbers

**Domain:**

- $x > 0$ (side length must be positive).
- $x < 12$ (squares cannot exceed half the shorter side).
- Domain: $x \in [0, 12]$.

**Find Critical Numbers:**

$$V'(x) = 12x^2 - 240x + 864.$$

Solve $V'(x) = 0$:

$$12x^2 - 240x + 864 = 0 \implies x^2 - 20x + 72 = 0.$$

Using the quadratic formula:

$$x = 10 \pm 2\sqrt{7}.$$

**Valid Critical Point:** $x = 10 - 2\sqrt{7} \approx 4.708.$

# Maximum Volume and Final Answer

**Maximum Volume:**

$$V(10-2\sqrt{7}) = 4(10-2\sqrt{7})^3 - 120(10-2\sqrt{7})^2 + 864(10-2\sqrt{7}).$$

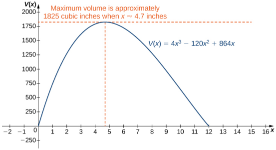
Approximation:

$$V \approx 1825\,\text{in}^3.$$

**Optimal Dimensions:**

- Height: $x \approx 4.708\,\text{in}$,
- Length: $36 - 2x \approx 26.584\,\text{in}$,
- Width: $24 - 2x \approx 14.584\,\text{in}$.

**Final Answer:** The maximum volume is approximately $1825\,\text{in}^3$ with optimal dimensions as listed above.



# Minimizing Travel Time

An island is located 2 mi due north of its closest point along a straight shoreline. A visitor is staying at a cabin on the shore, which is 6 mi west of that closest point. The visitor plans to travel from the cabin to the island.

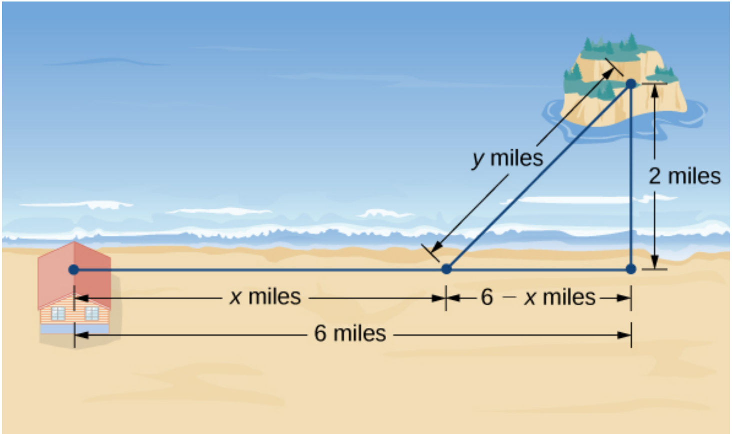
Suppose the visitor:

- Runs at a speed of 8 mph,
- Swims at a speed of 3 mph.

**Question:** How far should the visitor run along the shoreline before swimming to minimize the total time it takes to reach the island?

# Solution

Let $x$ be the distance running and let $y$ be the distance swimming . Let $T$ be the time it takes to get from the cabin to the island.



**Step 2:** The problem is to minimize $T$.

**Step 3:** To find the time spent traveling from the cabin to the island, add the time spent running and the time spent swimming. Since Distance $=$ Rate $\times$ Time, the time spent running is:

$$T_{\text{running}} = \frac{D_{\text{running}}}{R_{\text{running}}} = \frac{x}{8},$$

and the time spent swimming is:

$$T_{\text{swimming}} = \frac{D_{\text{swimming}}}{R_{\text{swimming}}} = \frac{y}{3}.$$

Therefore, the total time spent traveling is:

$$T = \frac{x}{8} + \frac{y}{3}.$$

**Step 4:** From (Figure), the line segment of $y$ miles forms the hypotenuse of a right triangle with legs of length 2 mi and $6 - x$ mi. Therefore, by the Pythagorean theorem:

$$2^2 + (6 - x)^2 = y^2,$$

and we obtain:

$$y = \sqrt{(6 - x)^2 + 4}.$$

# Solution

Thus, the total time spent traveling is given by the function:

$$T(x) = \frac{x}{8} + \frac{\sqrt{(6-x)^2+4}}{3}.$$

**Step 5:** From (Figure), we see that $0 \leq x \leq 6$. Therefore, $[0, 6]$ is the domain of consideration.

**Step 6**: Finding Critical Numbers

Since $T(x)$ is a continuous function over a closed, bounded interval, it has a maximum and a minimum. Let's begin by looking for any critical numbers of $T$ over the interval $[0, 6]$. The derivative is:

$$T'(x) = \frac{1}{8} - \frac{1}{2}\left[(6-x)^2+4\right]^{-1/2} \cdot 2(6-x) = \frac{1}{8} - \frac{(6-x)}{3\sqrt{(6-x)^2+4}}.$$

If $T'(x) = 0$, then:

$$\frac{1}{8} = \frac{6-x}{3\sqrt{(6-x)^2+4}}.$$

Therefore:

$$3\sqrt{(6-x)^2+4} = 8(6-x).$$

Squaring both sides of this equation, we see that if $x$ satisfies this equation, then $x$ must satisfy:

$$9\left[(6-x)^2+4\right] = 64(6-x)^2,$$

which implies:

$$55(6-x)^2 = 36.$$

We conclude that if $x$ is a critical number, then $x$ satisfies:

$$(6-x)^2 = \frac{36}{55}.$$

Therefore, the possibilities for critical numbers are:

$$x = 6 \pm \frac{6}{\sqrt{55}}.$$

Since $x = 6 + \frac{6}{\sqrt{55}}$ is not in the domain, it is not a possibility for a critical number. On the other hand, $x = 6 - \frac{6}{\sqrt{55}}$ is in the domain. Since we squared both sides to arrive at the possible critical numbers, it remains to verify that $x = 6 - \frac{6}{\sqrt{55}}$ satisfies the equation.
Since $x = 6 - \frac{6}{\sqrt{55}}$ does satisfy that equation, we conclude that it is the critical number.

$$x = 6 \pm \frac{6}{\sqrt{55}}.$$

Since $x = 6 + \frac{6}{\sqrt{55}}$ is not in the domain, it is not a possibility for a critical number. On the other hand, $x = 6 - \frac{6}{\sqrt{55}}$ is in the domain. Since we squared both sides to arrive at the possible critical numbers, it remains to verify that $x = 6 - \frac{6}{\sqrt{55}}$ satisfies the equation.

# Solution

Since $x = 6 - \frac{6}{\sqrt{55}}$ does satisfy that equation, we conclude that:

$$x = 6 - \frac{6}{\sqrt{55}}$$

is a critical number, and it is the only one. To justify that the time is minimized for this value of $x$, we just need to check the values of $T(x)$ at the endpoints $x = 0$ and $x = 6$, and compare them with the value of $T(x)$ at the critical number $x = 6 - \frac{6}{\sqrt{55}}$.

We find that:

$$T(0) \approx 2.108\,\text{h}, \quad T(6) \approx 1.417\,\text{h}, \quad \text{whereas} \quad T(6 - \frac{6}{\sqrt{55}}) \approx 1.368\,\text{h}.$$

Therefore, we conclude that $T$ has a local minimum at:

$$x \approx 5.19\,\text{mi}.$$

# Maximizing Revenue

Owners of a car rental company have determined that if they charge customers $p$ dollars per day to rent a car, where $50 \leq p \leq 200$, the number of cars $n$ they rent per day can be modeled by the linear function:

$$n(p) = 1000 - 5p.$$

- If they charge $50 perdayorless, theywillrentalltheircars. - Iftheycharge200$ per day or more, they will not rent any cars.

Assuming the owners plan to charge customers between $50 perdayand200$ per day to rent a car, how much should they charge to maximize their revenue?

# Step 1: Problem Setup

**Objective:** Maximize the daily revenue of a car rental company.

**Variables:**

- $p$: Price charged per car per day ($50 \leq p \leq 200$).
- $n$: Number of cars rented per day.
- $R$: Revenue per day.

**Model:**

- $n(p) = 1000 - 5p$: Linear function modeling the number of cars rented as a function of $p$.
- Revenue formula: $R = n \times p$.

# Step 2: Revenue Function

The revenue (per day) is given by:

$$R(p) = n(p) \times p.$$

Substituting $n(p) = 1000 - 5p$:

$$R(p) = (1000 - 5p)p = -5p^2 + 1000p.$$

**Goal:** Maximize $R(p)$ for $p$ in the interval $[50, 200]$.

# Step 3: Critical Numbers

To maximize $R(p)$, find its derivative and solve $R'(p) = 0$:

$$R'(p) = -10p + 1000.$$

Setting $R'(p) = 0$:

$$-10p + 1000 = 0 \quad \Longrightarrow \quad p = 100.$$

**Critical Point:** $p = 100$.

# Step 4: Evaluate Revenue at Endpoints

Evaluate $R(p)$ at the critical point and the endpoints of the interval $[50, 200]$:

- At $p = 100$:

$$R(100) = -5(100)^2 + 1000(100) = \$50,000.$$

- At $p = 50$:

$$R(50) = -5(50)^2 + 1000(50) = \$37,500.$$

- At $p = 200$:

$$R(200) = -5(200)^2 + 1000(200) = \$0.$$

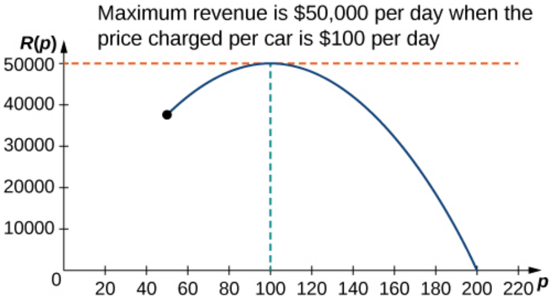
# Step 5: Conclusion

**Result:**

- The revenue is maximized at $p = 100$.
- Maximum revenue: $R(100) = \$50,000$.

**Recommendation:**

- The car rental company should charge \$100 per day per car to maximize daily revenue.



# Problem Statement

**Maximizing the Area of an Inscribed Rectangle**

A rectangle is to be inscribed in the ellipse:

$$\frac{x^2}{4} + y^2 = 1$$

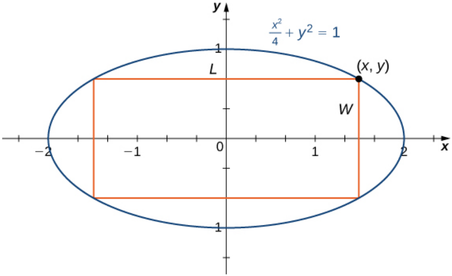
Determine:

- The dimensions of the rectangle that maximize its area.
- The maximum area.

# Step 1: Geometry of the Problem

- The ellipse $\frac{x^2}{4} + y^2 = 1$ has $x$-intercepts $\pm 2$ and $y$-intercepts $\pm 1$.
- An inscribed rectangle has:
  - Length $L = 2x$ (horizontal).
  - Width $W = 2y$ (vertical).
- The area of the rectangle is:

$$A = L \cdot W = 2x \cdot 2y = 4xy$$



# Step 2: Substituting Constraints

- From the ellipse equation:

$$\frac{x^2}{4} + y^2 = 1 \implies y = \sqrt{1 - \frac{x^2}{4}}$$

- Substitute $y$ into the area formula:

$$A = 4x\sqrt{1 - \frac{x^2}{4}} = 2x\sqrt{4 - x^2}$$

# Step 3: Domain and Critical Points

- $x \in [0, 2]$ (rectangle in the first quadrant).
- The derivative of $A(x)$ is:

$$A'(x) = \frac{d}{dx}\left(2x\sqrt{4 - x^2}\right)$$

Simplifying:

$$A'(x) = \frac{8 - 4x^2}{\sqrt{4 - x^2}}$$

- Solve $A'(x) = 0$ to find critical points:

$$8 - 4x^2 = 0 \implies x^2 = 2 \implies x = \sqrt{2}$$

# Step 4: Dimensions and Maximum Area

- At $x = \sqrt{2}$:

$$y = \sqrt{1 - \frac{x^2}{4}} = \sqrt{1 - \frac{2}{4}} = \frac{1}{\sqrt{2}}$$

- Dimensions of the rectangle:

$$L = 2x = 2\sqrt{2}, \quad W = 2y = \sqrt{2}$$

- Maximum area:

$$A = L \cdot W = (2\sqrt{2})(\sqrt{2}) = 4$$

# Optimization Example: Box Surface Area

**Problem:** Minimize the surface area of an open-top rectangular box with volume 216 $\text{in}^3$.

**Surface Area:**

$$S(x) = 4xy + x^2$$

**Volume Constraint:**

$$x^2 y = 216 \implies y = \frac{216}{x^2}$$

Substitute $y$ into $S(x)$:

$$S(x) = 864/x + x^2$$

# PSS: Justify a Maximum or Minimum on an Open Interval

**Step 1: Analyze the Limits**

- Take the limit of the function as the variable approaches the endpoints of the interval.
- If both limits are less than the function value at the critical number, the largest value at the critical points is the **absolute maximum** (similar for minimum).
- If at least one limit is larger (or smaller) than the critical values or diverges to infinity, then no maximum (or minimum) exists.

**Step 2: Check Monotonicity**

- Verify if the function is **increasing** to the left of the critical number and **decreasing** to the right (or vice versa).
- If true, the critical number corresponds to an **absolute maximum** (or minimum).

**Step 3: Verify with Critical Numbers**

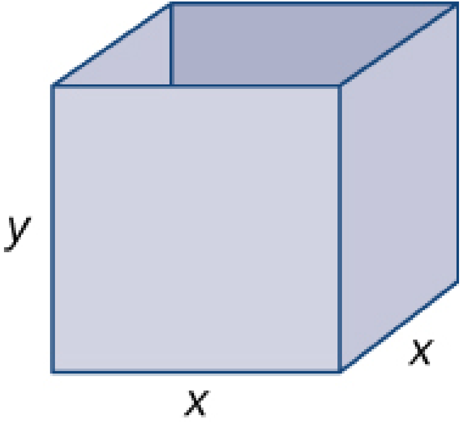
- If there is only **one critical number** on the interval, and the function has a **local maximum** (or minimum) at this value, it is also the **absolute maximum** (or minimum).

# Minimizing Surface Area

A rectangular box with a square base, an open top, and a volume of 216 in. 3 is to be constructed. What should the dimensions of the box be to minimize the surface area of the box? What is the minimum surface area?

# Solution: Step 1

**Step 1:** Draw a rectangular box and introduce the variable $x$ to represent the length of each side of the square base; let $y$ represent the height of the box. Let $S$ denote the surface area of the open-top box.



# Solution: Steps 2-4

**Step 2:** We need to minimize the surface area. Therefore, we need to minimize $S$.

**Step 3:** Since the box has an open top, we need only determine the area of the four vertical sides and the base.

- The area of each of the four vertical sides is $x \cdot y$.
- The area of the base is $x^2$.

Therefore, the surface area of the box is:

$$S = 4xy + x^2$$

**Step 4:** Since the volume of this box is $x^2y$ and the volume is given as $216\,\text{in.}^3$, the constraint equation is:

$$x^2y = 216$$

Solving the constraint equation for $y$, we have $y = \frac{216}{x^2}$. Therefore, we can write the surface area as a function of $x$ only:

$$S(x) = 4x\left(\frac{216}{x^2}\right) + x^2$$

**Step 5: Domain Analysis**

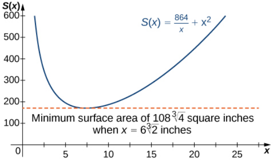
- Since $x^2y = 216$, $x > 0$ and $x$ is unbounded. Domain: $(0, \infty)$.
- As $x \to 0^+$ or $x \to \infty$, $S(x) \to \infty$. Therefore, $S(x)$ must have an absolute minimum on $(0, \infty)$.

**Step 6: Finding the Minimum**

- Derivative: $S'(x) = -\frac{864}{x^2} + 2x$.
- Solve $S'(x) = 0$: $x^3 = 432 \implies x = 6\sqrt[3]{2}$.
- At $x = 6\sqrt[3]{2}$, $y = \frac{216}{(6\sqrt[3]{2})^2} = 3\sqrt[3]{2}$.

**Results:**

$$x = 6\sqrt[3]{2}\,\text{in.},\;\; y = 3\sqrt[3]{2}\,\text{in.},\quad S = 108\sqrt[3]{4}\,\text{in.}^2$$



# Key Concepts for Solving Optimization Problems

1. **Draw a Picture:** Begin by drawing a diagram to visualize the problem and introducing variables to represent key quantities.
2. **Relate the Variables:** Find an equation that relates the variables in the problem.
3. **Define the Function:** Write the quantity to be minimized or maximized as a function of a single variable.
4. **Find Critical Numbers:** Compute the derivative, find critical numbers, and determine the local extrema.

# Curve Sketching (omit oblique asymptotes)

Clotilde Djuikem

# Learning Objectives

- **Objective 1:** Analyze a function and its derivatives to draw its graph.
- **Objective 2:** Integrate the use of the first and second derivatives with other features of a function to create an accurate graph of $f(x)$.

**Description:**
In this section, we explore a comprehensive approach to graphing functions. By combining knowledge of derivatives with other analytical tools, you will be able to sketch the shape and key features of any given function.

# Problem-Solving Strategy: Drawing the Graph of a Function

**Steps:**

1. **Determine the Domain:** Identify all $x$-values for which the function is defined.
2. **Intercepts:** Locate $x$- and $y$-intercepts.
3. **End Behavior:** Evaluate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ to find horizontal or oblique asymptotes.
4. **Vertical Asymptotes:** Check for $x$-values where the function approaches $\pm\infty$.
5. **First Derivative Analysis ($f'(x)$):**
   - Identify critical points.
   - Determine intervals of increase and decrease.
   - Locate local extrema.
6. **Second Derivative Analysis ($f''(x)$):**
   - Determine concavity (up or down).
   - Find inflection points.
   - Confirm or verify extrema.

# Sketching a Graph of a Polynomial

**Example: Sketch a graph of** $f(x) = (x-1)^2(x+2)$

# Step 1: Determine the Domain

**Function:** $f(x) = (x-1)^2(x+2)$

- $f(x)$ is a polynomial, so it is defined for all real numbers.
- **Domain:** $\mathbb{R}$ (all real numbers).

## Step 2: Locate the Intercepts

- *y*-**Intercept:**

$$f(0) = (0-1)^2(0+2) = 2 \quad \Longrightarrow \text{Intercept: } (0,2).$$

- *x*-**Intercepts:** Solve $(x-1)^2(x+2) = 0$:

$$x = 1 \quad \text{(multiplicity 2)}, \quad x = -2.$$

- **Intercept Points:** $(1,0), (-2,0)$.

# Step 3: End Behavior

Evaluate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$:

- As $x \to \infty$:

$$(x-1)^2 \to \infty, \quad (x+2) \to \infty \implies \lim_{x\to\infty} f(x) = \infty.$$

- As $x \to -\infty$:

$$(x-1)^2 \to \infty, \quad (x+2) \to -\infty \implies \lim_{x\to-\infty} f(x) = -\infty.$$

# Step 4: Vertical Asymptotes

**Observation:**

- $f(x)$ is a polynomial function.
- Polynomial functions do not have vertical asymptotes.
- **Conclusion:** No vertical asymptotes exist for $f(x)$.

# Step 5: First Derivative Analysis

**First Derivative:**

$$f'(x) = 3x^2 - 3 = 3(x-1)(x+1).$$

**Critical Numbers:** Solve $f'(x) = 0$:

$$x = 1, \quad x = -1.$$

Divide the domain into intervals and test $f'(x)$:

- $(-\infty, -1) : f'(x) > 0 \quad$ (increasing).
- $(-1, 1) : f'(x) < 0 \quad$ (decreasing).
- $(1, \infty) : f'(x) > 0 \quad$ (increasing).

**Local Extrema:**

$$f(-1) = 4 \quad \text{(local max)}, \quad f(1) = 0 \quad \text{(local min)}.$$

# Step 6: Second Derivative and Concavity

**Second Derivative:**

$$f''(x) = 6x.$$

**Concavity Analysis:**

- $(-\infty, 0) : f''(x) < 0 \quad \text{(concave down)}.$
- $(0, \infty) : f''(x) > 0 \quad \text{(concave up)}.$

Inflection point at $x = 0$.

**Summary of Key Points:**

- Intervals of increase: $(-\infty, -1), (1, \infty)$.
- Intervals of decrease: $(-1, 1)$.
- Local maximum: $(-1, 4)$, local minimum: $(1, 0)$.
- Concave up: $(0, \infty)$, concave down: $(-\infty, 0)$.
- Inflection point: $(0, f(0)) = (0, 2)$.

# Graph of $f(x) = (x-1)^2(x+2)$

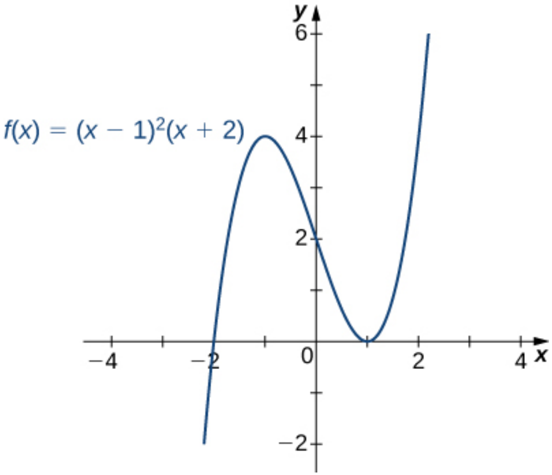



Figure:

# Exercice

Sketching the Graph of $f(x) = (x-1)^3(x+2)$

# Step 1: Determine the Domain

**Function:** $f(x) = (x-1)^3(x+2)$

- $f(x)$ is a polynomial, so it is defined for all real numbers.
- **Domain:** $\mathbb{R}$ (all real numbers).

# Step 2: Locate the Intercepts

- $y$-**Intercept:**

$$f(0) = (0-1)^3(0+2) = (-1)^3(2) = -2.$$

$y$-**Intercept:** $(0, -2)$**.**

- $x$-**Intercepts:** Solve $(x-1)^3(x+2) = 0$:

$$x = 1 \quad \text{(multiplicity 3)}, \quad x = -2 \quad \text{(multiplicity 1)}.$$

$x$-**Intercepts:** $(1, 0), (-2, 0)$**.**

# Step 3: Evaluate End Behavior

- The degree of $f(x)$ is 4 (even degree), and the leading coefficient is positive.
- As $x \to \infty$:

$$(x-1)^3 \to \infty, \quad (x+2) \to \infty \implies \lim_{x \to \infty} f(x) = \infty.$$

- As $x \to -\infty$:

$$(x-1)^3 \to -\infty, \quad (x+2) \to -\infty \implies \lim_{x \to -\infty} f(x) = \infty.$$

- **Conclusion:** Both ends of the graph go to $\infty$.

# Step 4: Check for Vertical Asymptotes

**Observation:**

- $f(x)$ is a polynomial function.
- Polynomial functions do not have vertical asymptotes.
- **Conclusion:** No vertical asymptotes exist for $f(x)$.

# Step 5: Analyze $f'(x)$

**First Derivative:**

$$f'(x) = 3(x-1)^2(x+2) + (x-1)^3 = (x-1)^2(4x+5).$$

**Critical Numbers:** Solve $f'(x) = 0$:

- $(x-1)^2 = 0 \implies x = 1.$
- $4x + 5 = 0 \implies x = -\frac{5}{4}.$

Divide the domain into intervals and test the sign of $f'(x)$, the derivative signed change only with $4x - 5$ because $(x-1)^2 \geq 0$:

- $(-\infty, -\frac{5}{4}) : f'(x) > 0 \quad \text{(decreasing)}.$
- $(-\frac{5}{4}, 1) : f'(x) > 0 \quad \text{(increasing)}.$
- $(1, \infty) : f'(x) > 0 \quad \text{(increasing)}.$

# Step 6: Analyze $f''(x)$

**Second Derivative:**

$$f''(x) = 6(x-1)(2x+1).$$

**Concavity Analysis:**

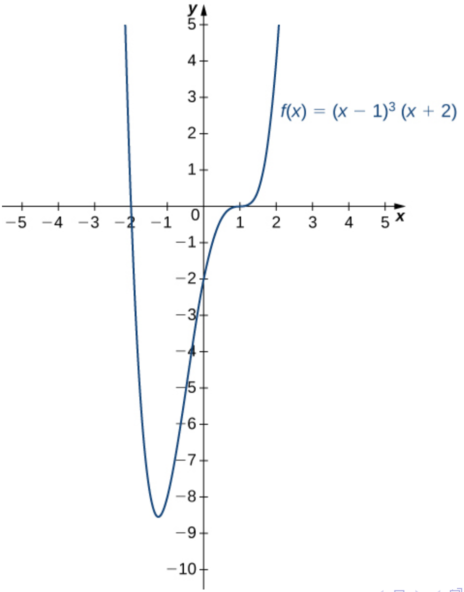
- $(-\infty, -\frac{1}{2}) : f''(x) > 0 \quad \text{(concave up)}.$
- $(-\frac{1}{2}, 1) : f''(x) < 0 \quad \text{(concave down)}.$
- $(1, \infty) : f''(x) > 0 \quad \text{(concave up)}.$

Inflection pointx at $x = -\frac{1}{2}$ and $x = 1$.

**Summary of Key Points:**

- Intervals of increase: $(-\frac{5}{4}, \infty)$.
- Intervals of decrease: $(-\infty, -\frac{5}{4})$.
- Local minimum: At $x = -\frac{5}{4}$
- Concave up: $(-\infty, -\frac{1}{2})$, $(1, \infty)$ concave down: $(-\infty, 0)$.
- Inflection points: $(-\frac{1}{2}, f(-\frac{1}{2}))$ and $(1, f(1))$.

# Graph of $f(x) = (x-1)^3(x+2)$



# Approximating Areas

Clotilde Djuikem

# Learning Objectives

- Use sigma (summation) notation to calculate sums and powers of integers.
- Use the sum of rectangular areas to approximate the area under a curve.
- Use Riemann sums to approximate area.

# Definitions

**Proof**

Let $x$ such that $f(x) = x^2$

**Example**

Consider the function $f(x) = \frac{x}{x^2+2}$

# Antiderivatives

Clotilde Djuikem

# Learning Objectives

- **Find the general antiderivative of a given function.**
  - Learn to determine the most general form of a function's antiderivative.
- **Explain the terms and notation used for an indefinite integral.**
  - Understand key concepts such as the integral sign ($\int$) and the constant of integration ($+C$).
- **State the power rule for integrals.**
  - Master the rule: $\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$, where $n \neq -1$.
- **Use antidifferentiation to solve simple initial-value problems.**
  - Apply integration techniques to find specific solutions when initial conditions are provided.

# Definition and Derivation

## Definition

A function $F$ is an antiderivative of the function $f$ if

$$F'(x) = f(x)$$

for all $x$ in the domain of $f$.

## Derivation Example

Let $f(x) = 2x$. To find the antiderivative $F(x)$, we solve:

$$F'(x) = f(x) = 2x.$$

By integrating, we obtain:

$$F(x) = \int 2x\,dx = x^2,\; F(x) = x^2 - 2,\; F(x) = x^2 - 10$$

# General Form of an Antiderivative

## Definition

Let $F$ be an antiderivative of $f$ over an interval $I$. Then:

1. For each constant $C$, the function $F(x) + C$ is also an antiderivative of $f$ over $I$.
2. If $G$ is an antiderivative of $f$ over $I$, there is a constant $C$ for which $G(x) = F(x) + C$ over $I$.

## Key Result

In other words, the most general form of the antiderivative of $f$ over $I$ is:

$$F(x) + C.$$

# Functions and Their Antiderivatives

| **Function** $f(x)$ | **Antiderivative** $F(x)$ |
|---|---|
| $4x^3$ | $x^4 + C$ |
| $\cos(x)$ | $\sin(x) + C$ |
| $e^x$ | $e^x + C$ |
| $\frac{1}{x}$ | $\ln\|x\| + C$ |
| $\sin(x)$ | $-\cos(x) + C$ |
| $5$ | $5x + C$ |
| $x^{-2}$ | $-\frac{1}{x} + C$ |

# Definition

## Definition

Given a function $f$, the indefinite integral of $f$, denoted

$$\int f(x)\,dx,$$

is the most general antiderivative of $f$. If $F$ is an antiderivative of $f$, then:

$$\int f(x)\,dx = F(x) + C.$$

The expression $f(x)$ is called the *integrand*, and the variable $x$ is the *variable of integration*.

# Examples

## Example

Let $f(x) = 3x^2$. To find its general antiderivative, solve:

$$F'(x) = f(x) = 3x^2.$$

Integrating, we obtain:

$$F(x) = \int 3x^2\,dx = x^3 + C.$$

Thus, the general form of the antiderivative is $F(x) + C = x^3 + C$, where $C$ is a constant.

## Example 1: Polynomial Function

Find the general antiderivative of $f(x) = 4x^3$.

$$F(x) = \int 4x^3\,dx = \frac{4x^4}{4} + C = x^4 + C.$$

# Examples

## Example 2: Trigonometric Function

Find the general antiderivative of $f(x) = \cos(x)$.

$$F(x) = \int \cos(x)\, dx = \sin(x) + C.$$

## Example 3: Exponential Function

Find the general antiderivative of $f(x) = e^x$.

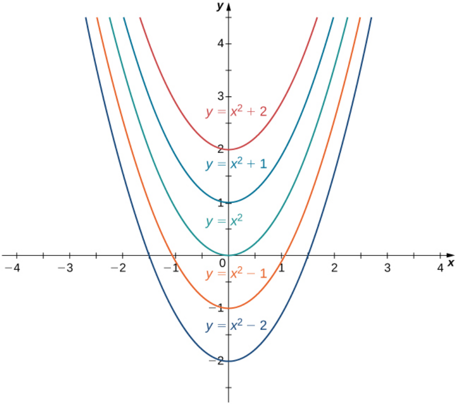
$$F(x) = \int e^x\, dx = e^x + C.$$

## Example 4: Reciprocal Function

Find the general antiderivative of $f(x) = \frac{1}{x}$ for $x > 0$.

$$F(x) = \int \frac{1}{x}\, dx = \ln(x) + C.$$

$$\int 2x\,dx = x^2 + C.$$




For different values of $C$

# Differentiation Formulas and Indefinite Integrals

| Differentiation Formula | Indefinite Integral |
|---|---|
| $\frac{d}{dx}(k) = 0$ | $\int k\,dx = kx + C$ |
| $\frac{d}{dx}(x^n) = nx^{n-1}$ | $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1$ |
| $\frac{d}{dx}(\ln\lvert x\rvert) = \frac{1}{x}$ | $\int \frac{1}{x}\,dx = \ln\lvert x\rvert + C$ |
| $\frac{d}{dx}(e^x) = e^x$ | $\int e^x\,dx = e^x + C$ |
| $\frac{d}{dx}(\sin x) = \cos x$ | $\int \cos x\,dx = \sin x + C$ |
| $\frac{d}{dx}(\csc x) = -\csc x \cot x$ | $\int \csc x \cot x\,dx = -\csc x + C$ |
| $\frac{d}{dx}(\sec x) = \sec x \tan x$ | $\int \sec x \tan x\,dx = \sec x + C$ |
| $\frac{d}{dx}(\cos x) = -\sin x$ | $\int \sin x\,dx = -\cos x + C$ |
| $\frac{d}{dx}(\tan x) = \sec^2 x$ | $\int \sec^2 x\,dx = \tan x + C$ |
| $\frac{d}{dx}(\cot x) = -\csc^2 x$ | $\int \csc^2 x\,dx = -\cot x + C$ |
| $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$ | $\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$ |
| $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$ | $\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C$ |
| $\frac{d}{dx}(\sec^{-1}\lvert x\rvert) = \frac{1}{x\sqrt{x^2-1}}$ | $\int \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1}\lvert x\rvert + C$ |

# Properties of Indefinite Integrals

## Properties

Let $F$ and $G$ be antiderivatives of $f$ and $g$, respectively, and let $k$ be any real number.

**Sums and Differences**

$$\int (f(x) \pm g(x))dx = F(x) \pm G(x) + C$$

**Constant Multiples**

$$\int kf(x)dx = kF(x) + C$$

# Solving an Initial-Value Problem

**Problem:** Solve the initial-value problem:

$$\frac{dy}{dx} = \sin x, \quad y(0) = 5.$$

**Solution:**

- First, solve the differential equation. If

$$\frac{dy}{dx} = \sin x,$$

then

$$y = \int \sin(x) dx = -\cos x + C.$$

- Apply the initial condition $y(0) = 5$. Substituting:

$$-\cos(0) + C = 5.$$

- Solve for $C$:

$$C = 5 + \cos(0) = 6.$$

**Final Solution:** $y = -\cos x + 6.$

# Solving an Initial Value Problem

**Problem:** Solve the initial value problem:

$$\frac{dy}{dx} = 3x^{-2}, \quad y(1) = 2.$$

**Solution:**

- Start by solving the differential equation. Integrate both sides:

$$y = \int 3x^{-2} dx = 3 \int x^{-2} dx = 3\left(-x^{-1}\right) + C.$$

  Thus,

$$y = -\frac{3}{x} + C.$$

- Use the initial condition $y(1) = 2$ to find $C$:

$$2 = -\frac{3}{1} + C \implies C = 2 + 3 = 5.$$

- Therefore, the solution to the initial value problem is:

$$y = -\frac{3}{x} + 5.$$

# Decelerating Car

A car is traveling at the rate of 88 ft/sec (60 mph) when the brakes are applied. The car begins decelerating at a constant rate of 15 $\text{ft}/\text{sec}^2$.

a. How many seconds elapse before the car stops?
b. How far does the car travel during that time?

# Decelerating Car Solution

**Solution:**

a. **Time to Stop:**

- Let $t$ represent time (in seconds), $a(t)$ be the acceleration, $v(t)$ the velocity, and $s(t)$ the position of the car. The initial velocity is $v(0) = 88\,\text{ft/sec}$, and the deceleration is constant: $a(t) = -15\,\text{ft/sec}^2$.
- From $v'(t) = a(t) = -15$, solve the initial-value problem:

$$v'(t) = -15, \quad v(0) = 88.$$

- Integrate:

$$v(t) = \int -15\,dt = -15t + C.$$

- Using $v(0) = 88$, solve for $C$: $C = 88$, so:

$$v(t) = -15t + 88.$$

- The car stops when $v(t) = 0$:

$$0 = -15t + 88 \implies t = \frac{88}{15} \approx 5.87\,\text{seconds}.$$

# Decelerating Car Solution

**Solution:**

b. **Distance Traveled:**

- The velocity $v(t)$ is the derivative of the position $s(t)$. Solve the initial-value problem:

$$s'(t) = v(t) = -15t + 88, \quad s(0) = 0.$$

- Integrate:

$$s(t) = \int (-15t + 88)\, dt = -\frac{15}{2}t^2 + 88t + C.$$

- Using $s(0) = 0$, solve for $C$: $C = 0$, so:

$$s(t) = -\frac{15}{2}t^2 + 88t.$$

- Evaluate $s(t)$ at $t = \frac{88}{15}$:

$$s\left(\frac{88}{15}\right) = -\frac{15}{2}\left(\frac{88}{15}\right)^2 + 88\left(\frac{88}{15}\right) = \frac{7744}{30} \approx 258.133\ ft.$$

# Stopping Distance Problem

Suppose the car is traveling at the rate of 44 ft/sec.

1. How long does it take for the car to stop?
2. How far will the car travel during this time?

# Gravity

A rock is dropped from the top of a building 10 metres above the ground on Earth.

a. How long until the rock hits the ground?

b. With what velocity does the rock hit the ground?

# Key Concepts

## Key Concepts

- If $F$ is an antiderivative of $f$, then every antiderivative of $f$ is of the form $F(x) + C$ for some constant $C$.
- Solving the initial-value problem

$$\frac{dy}{dx} = f(x), \quad y(x_0) = y_0$$

requires us first to find the set of antiderivatives of $f$ and then to look for the particular antiderivative that also satisfies the initial condition.