Defining the Derivative

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Why Do We Talk About Speed?

Definition of Speed

Speed measures the change in an object's position over time. It is the derivative of position with respect to time.

Example: If a car covers a distance of 100 km in 2 hours, the average speed is:

$$\text{Speed} = \frac{\text{Distance}}{\text{Time}} = \frac{100 \text{ km}}{2 \text{ h}} = 50 \text{ km/h}$$

This shows how position changes over time. But how does the speed change at every instant? That's where derivatives come in.

Derivative and Instantaneous Speed

The Derivative:

The derivative measures how one quantity changes with respect to another. The derivative of position with respect to time gives the **instantaneous speed**, i.e., the speed at each exact moment.

Example: If the position $s(t)$ of a car is given by $s(t) = 5t^2$, the derivative $s'(t)$ is:

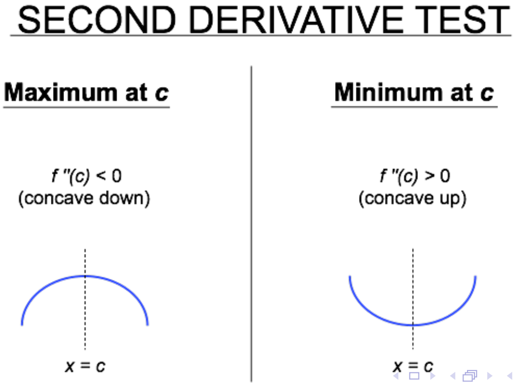
$$s'(t) = \frac{ds}{dt} = 10t$$

This means at any time t, the speed of the car is $10t$ km/h.

Application of Derivatives in Optimization

Derivatives are essential for optimizing functions.

- **Finding Minima and Maxima:** By calculating the derivative of a function and setting it to zero, we find the points where the function reaches its minimum or maximum.
- **Example:** For a curve representing the production cost of a product, the derivative helps find the minimum cost.



Derivatives and Neural Networks Learning

Why Derivatives in AI?

In neural networks, derivatives are used to minimize the error between the prediction and reality. This process is called backpropagation.

- **Loss Function:** The derivative of the loss function with respect to the model parameters (weights and biases) tells us how to adjust them to reduce the error.
- **Gradient Descent:** The gradient descent method uses derivatives to find the minimum of the loss function, where the error is the smallest.

Learning Objectives

- Recognize the meaning of the tangent to a curve at a point.
- Calculate the slope of a tangent line.
- Identify the derivative as the limit of a difference quotient.
- Calculate the derivative of a given function at a point.
- Describe the velocity as a rate of change.
- Explain the difference between average velocity and instantaneous velocity.
- Estimate the derivative from a table of values.

Definition

Let f be a function defined on an interval I containing a. If $x \neq a$ is in I, then

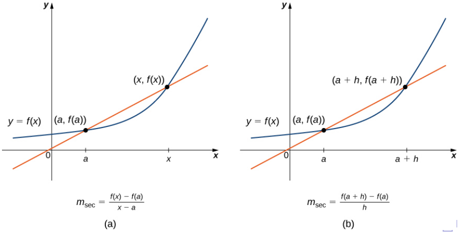
$$Q = \frac{f(x) - f(a)}{x - a}$$

is a **difference quotient**.

Also, if $h \neq 0$ is chosen so that $a + h$ is in I, then

$$Q = \frac{f(a+h) - f(a)}{h}$$

is a difference quotient with increment h.



Definition

Let $f(x)$ be a function defined in an open interval containing a. The tangent line to $f(x)$ at a is the line passing through the point $(a, f(a))$ having slope

$$m_{\tan} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided this limit exists.

Alternative Definition

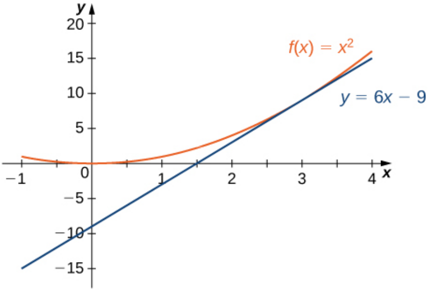
Equivalently, we may define the tangent line to $f(x)$ at a to be the line passing through the point $(a, f(a))$ having slope

$$m_{\tan} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

provided this limit exists.

Finding a Tangent Line

Find the equation of the line tangent to the graph of $f(x) = x^2$ at $x = 3$.

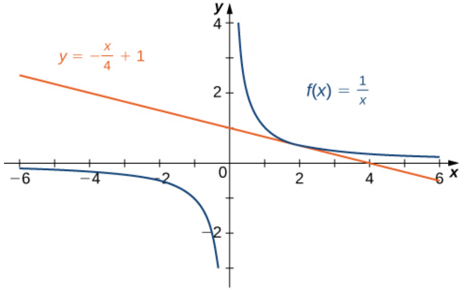


The Slope of a Tangent Line Revisited

Use (Figure) to find the slope of the line tangent to the graph of $f(x) = x^2$ at $x = 3$.

Finding the Equation of a Tangent Line

Find the equation of the line tangent to the graph of $f(x) = \frac{1}{x}$ at $x = 2$.



Finding the Slope of a Tangent Line

Find the slope of the line tangent to the graph of $f(x) = \sqrt{x}$ at $x = 4$.

Definition

Let $f(x)$ be a function defined in an open interval containing a. The derivative of the function $f(x)$ at a, denoted by $f'(a)$, is defined by

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided this limit exists.

Alternative Definition

Alternatively, we may also define the derivative of $f(x)$ at a as

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

provided this limit exists.

Estimating a Derivative

For $f(x) = x^2$, use a table to estimate $f'(3)$.

Solution

Create a table using values of x just below 3 and just above 3.

x	$\frac{x^2-9}{x-3}$
2.9	5.9
2.99	5.99
2.999	5.999
3.001	6.001
3.01	6.01
3.1	6.1

After examining the table, we see that a good estimate is $f'(3) = 6$.

Estimating a Derivative

For $f(x) = x^2$, use a table to estimate $f'(3)$.

Hint

Evaluate

$$\frac{(x+h)^2 - x^2}{h}$$

at $h = -0.1, -0.01, -0.001, 0.001, 0.01, 0.1$.

Solution

Using $x = 3$ and evaluating the difference quotient for different values of h:

h	$\frac{(3+h)^2-9}{h}$
-0.1	5.9
-0.01	5.99
-0.001	5.999
0.001	6.001
0.01	6.01

Finding a Derivative

For $f(x) = 3x^2 - 4x + 1$, find $f'(2)$ by using (Figure).

Finding a Derivative

For $f(x) = x^2 + 3x + 2$, find $f'(1)$.

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Average and Instantaneous Velocity

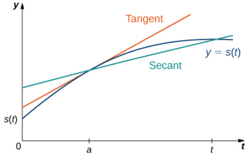
If $s(t)$ is the position of an object over time, the average velocity over the interval $[a, t]$ is:

$$v_{\text{avg}} = \frac{s(t) - s(a)}{t - a}$$

As t approaches a, the average velocity approaches the instantaneous velocity at a, given by:

$$v(a) = s'(a) = \lim_{t \to a} \frac{s(t) - s(a)}{t - a}$$

The slope of the tangent line represents instantaneous velocity, while the slope of the secant line represents average velocity over the interval $[a, t]$.



Definition

Instantaneous Rate of Change

The **instantaneous rate of change** of a function $f(x)$ at a value a is its derivative $f'(a)$.

Instantaneous Rate of Change of Temperature

A homeowner sets the thermostat so that the temperature in the house begins to drop from $70^\circ F$ at 9 p.m., reaches a low of $60^\circ F$ during the night, and rises back to $70^\circ F$ by 7 a.m. Suppose the temperature in the house is given by $T(t) = 0.4t^2 - 4t + 70$ for $0 \le t \le 10$, where t is the number of hours past 9 p.m. Find the instantaneous rate of change of the temperature at midnight.

Solution

Since midnight is 3 hours past 9 p.m., we want to compute $T'(3)$.

$$T'(3) = \lim_{t \to 3} \frac{T(t) - T(3)}{t - 3}$$

Substitute $T(t) = 0.4t^2 - 4t + 70$ and $T(3) = 61.6$:

$$T'(3) = \lim_{t \to 3} \frac{0.4t^2 - 4t + 70 - 61.6}{t - 3} = \lim_{t \to 3} \frac{0.4(t-3)(t-7)}{t-3}$$

Simplify and cancel:

$$T'(3) = 0.4(t-7) \quad \text{at } t = 3$$

$$T'(3) = -1.6$$

Rate of Change of Profit

A toy company can sell x electronic gaming systems at a price of $p = -0.01x + 400$ dollars per gaming system. The cost of manufacturing x systems is given by $C(x) = 100x + 10,000$ dollars. Find the rate of change of profit when 10,000 games are produced. Should the toy company increase or decrease production?

Solution

The profit $P(x)$ earned by producing x gaming systems is $R(x) - C(x)$, where $R(x)$ is the revenue obtained from selling x games.
$R(x) = xp = x(-0.01x + 400) = -0.01x^2 + 400x$
Thus, the profit function is: $P(x) = -0.01x^2 + 300x - 10,000$
To find the rate of change of profit when 10,000 games are produced, we compute $P'(10,000)$:

$$P'(10,000) = \lim_{x \to 10,000} \frac{P(x) - P(10,000)}{x - 10,000}$$

Substituting into the profit function and simplifying:

$$P'(10,000) = 100$$

Key Concepts

- The slope of the tangent line to a curve measures the instantaneous rate of change of a curve. We can calculate it by finding the limit of the difference quotient or the difference quotient with increment h.
- The derivative of a function $f(x)$ at a value a is found using either of the definitions for the slope of the tangent line.
- Velocity is the rate of change of position. As such, the velocity $v(t)$ at time t is the derivative of the position $s(t)$ at time t. Average velocity is given by:

$$v_{\text{avg}} = \frac{s(t) - s(a)}{t - a}$$

Instantaneous velocity is given by:

$$v(a) = s'(a) = \lim_{t \to a} \frac{s(t) - s(a)}{t - a}$$

- We may estimate a derivative by using a table of values.

The Derivative as a Function

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Learning Objectives

- Define the derivative function of a given function.
- Graph a derivative function from the graph of a given function.
- State the connection between derivatives and continuity.
- Describe three conditions for when a function does not have a derivative.
- Explain the meaning of a higher-order derivative.

Derivative Functions

The derivative function gives the derivative of a function at each point in the domain of the original function for which the derivative is defined. The formal definition is:

Definition

Let $f(x)$ be a function. The derivative function, denoted by $f'(x)$, is defined as:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

A function is said to be differentiable at $x = a$ if $f'(a)$ exists.

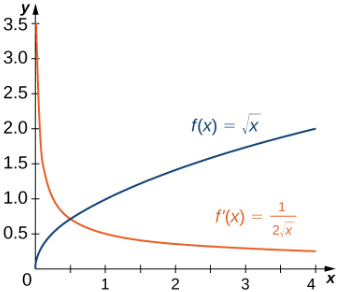
Finding the Derivative of a Square-Root Function

Find the derivative of $f(x) = \sqrt{x}$.
Solution:

$$f'(x) = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} = \lim_{h\to 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})} = \frac{1}{2\sqrt{x}}$$

Then

$$f'(x) = (\sqrt{x})' = \frac{1}{2\sqrt{x}}$$



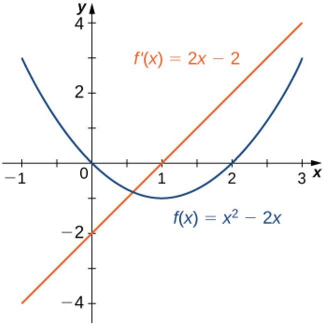
Finding the Derivative of a Quadratic Function

Find the derivative of $f(x) = x^2 - 2x$.

Solution:

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - 2(x+h) - (x^2 - 2x)}{h}$$

$$= \lim_{h \to 0} \frac{2xh - 2h + h^2}{h} = 2x - 2$$



Notations for Derivatives

We use different notations to represent the derivative of a function $f(x)$:

- $f'(x)$
- $\frac{dy}{dx}$
- y'
- $\frac{d}{dx}f(x)$

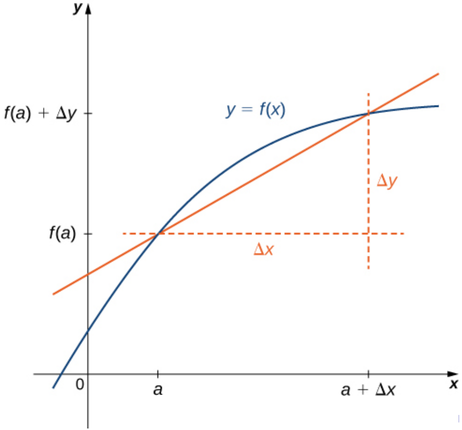
The derivative at a specific point $x = a$ can also be written as:

$$\left.\frac{dy}{dx}\right|_{x=a}$$

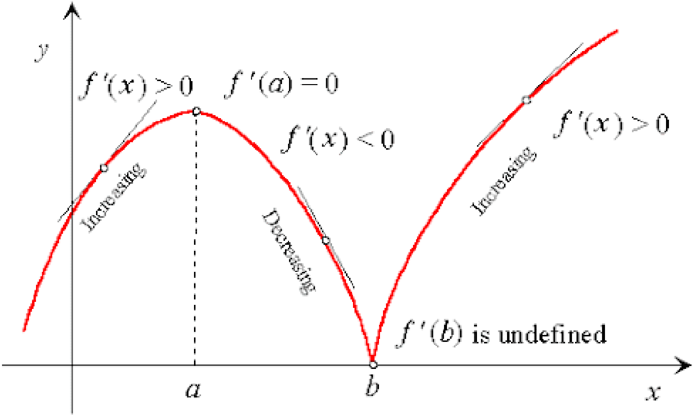
Graph derivative

The derivative as the instantaneous rate of change of a function y with respect to x is given by

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

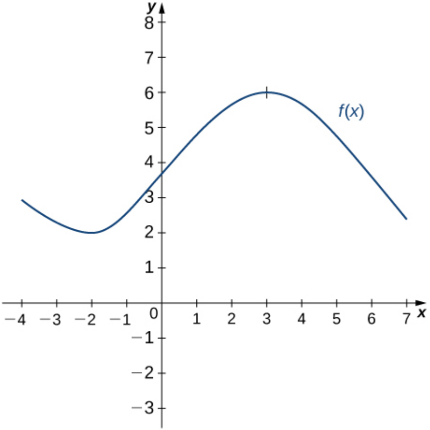


The graph of the derivative function $f'(x)$ is related to the graph of $f(x)$. If $f(x)$ is increasing, $f'(x) > 0$, and if $f(x)$ is decreasing, $f'(x) < 0$.



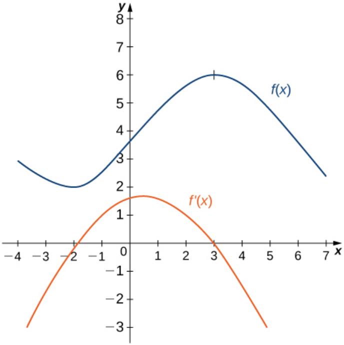
Exercise: Sketching a Derivative Using a Function

Use the following graph of $f(x)$ to sketch a graph of $f'(x)$.



Solution

- **Critical Points**: - At $x = -2$ (local minimum), $f'(x) = 0$. - At $x = 3$ (local maximum), $f'(x) = 0$.
- **Behavior of the Function**: - For $x < -2$, $f(x)$ is decreasing, so $f'(x)$ is negative. - For $-2 < x < 3$, $f(x)$ is increasing, so $f'(x)$ is positive. - For $x > 3$, $f(x)$ is decreasing again, so $f'(x)$ is negative.
- **Crossing Points**: - $f'(x)$ crosses the x-axis at $x = -2$ and $x = 3$, where the slope of $f(x)$ is zero.



Differentiability Implies Continuity

Theorem: If $f(x)$ is differentiable at $x = a$, then $f(x)$ is continuous at $x = a$.

Proof:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

This shows that:

$$\lim_{x \to a} f(x) = f(a)$$

Hence, $f(x)$ is continuous at $x = a$.

Important

If a function is not continuous, it cannot be differentiable, since every differentiable function must be continuous. However, if a function is continuous, it may still fail to be differentiable

Example: Continuity Does Not Imply Differentiability

We have just proven that differentiability implies continuity, but does continuity imply differentiability? Let's explore this question with the function $f(x) = |x|$.

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{|x|}{x}$$

This limit does not exist because:

$$\lim_{x \to 0^-} \frac{|x|}{x} = -1 \quad \text{and} \quad \lim_{x \to 0^+} \frac{|x|}{x} = 1$$

Since the left-hand limit and right-hand limit are not equal, the derivative at $x = 0$ does not exist, even though $f(x) = |x|$ is continuous at $x = 0$. Therefore, continuity does not necessarily imply differentiability.

Continuous and Differentiable Function at $x = 3$

Given the function:

$$f(x) = \begin{cases} ax + b & \text{if } x < 3 \\ x^2 & \text{if } x \geq 3 \end{cases}$$

We need to find values of a and b such that $f(x)$ is continuous and differentiable at $x = 3$.

Step 1: Continuity at $x = 3$

For continuity, we require: $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x)$ Substitute the function: $3a + b = 9$

Step 2: Differentiability at $x = 3$

For differentiability, the derivatives from both sides must be equal:

$$\lim_{x\to 3^-} f'(x) = \lim_{x\to 3^+} f'(x)$$

Differentiate both sides: $a = 6$ Substitute $a = 6$ into the continuity equation to find b:

$$3(6) + b = 9 \quad \Rightarrow \quad b = -9$$

Thus, the values of a and b are $a = 6$ and $b = -9$.

A Piecewise Function that is Continuous and Differentiable

A toy company wants to design a track for a toy car that starts out along a parabolic curve and then converts to a straight line. The function that describes the track is to have the form:

$$f(x) = \begin{cases} \frac{1}{10}x^2 + bx + c & \text{if } x < -10 \\ -\frac{1}{4}x + \frac{5}{2} & \text{if } x \geq -10 \end{cases}$$

For the car to move smoothly along the track, the function $f(x)$ must be both continuous and differentiable at $x = -10$. Find values of b and c that make $f(x)$ both continuous and differentiable.

Higher-Order Derivatives

Higher-order derivatives are the derivatives of derivatives. For example, the second derivative $f''(x)$ is the derivative of $f'(x)$. The notation can be:

$$f''(x), f^{(3)}(x), \ldots, f^{(n)}(x)$$

Or in Leibniz notation:

$$\frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \ldots$$

Finding Higher-Order Derivatives

Find the second derivative of $f(x) = 2x^2 - 3x + 1$.

Solution:

$$f'(x) = 4x - 3$$

$$f''(x) = 4$$

Finding a Second Derivative

For $f(x) = 2x^2 - 3x + 1$, find $f''(x)$.

Solution:

- First, find $f'(x)$:

$$f'(x) = \lim_{h \to 0} \frac{(2(x+h)^2 - 3(x+h) + 1) - (2x^2 - 3x + 1)}{h}$$

$$= \lim_{h \to 0} \frac{4xh + h^2 - 3h}{h}$$

$$= \lim_{h \to 0} (4x + h - 3)$$

$$= 4x - 3$$

- Next, find $f''(x)$ by taking the derivative of $f'(x) = 4x - 3$:

$$f''(x) = 4$$

Finding Acceleration

The position of a particle along a coordinate axis at time t (in seconds) is given by

$$s(t) = 3t^2 - 4t + 1 \quad \text{(in meters)}.$$

Find the function that describes its acceleration at time t.

Solution:

- Since $v(t) = s'(t)$ and $a(t) = v'(t) = s''(t)$, we begin by finding the derivative of $s(t)$:

$$s'(t) = \lim_{h \to 0} \frac{s(t+h) - s(t)}{h} = \lim_{h \to 0} \frac{3(t+h)^2 - 4(t+h) + 1 - (3t^2 - 4t}{h}$$

- Next, find $s''(t)$:

$$s''(t) = \lim_{h \to 0} \frac{s'(t+h) - s'(t)}{h} = \lim_{h \to 0} \frac{(6(t+h) - 4) - (6t - 4)}{h} = 6.$$

Thus, $a = 6\,\mathrm{m/s^2}$.

Finding Acceleration

For $s(t) = t^3$, find $a(t)$.
Solution:

- First, find the velocity function by taking the derivative of $s(t)$:

$$v(t) = s'(t) = 3t^2.$$

- Next, find the acceleration function by taking the derivative of $v(t)$:

$$a(t) = v'(t) = s''(t) = 6t.$$

Thus, $a(t) = 6t$.

Key Concepts

- The derivative of a function $f(x)$ represents the slope of the tangent line to the function at each point.
- Differentiability implies continuity, but continuity does not always imply differentiability.
- Higher-order derivatives represent successive rates of change, such as velocity and acceleration.

Differentiation Rules

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Learning Objectives

- **Identify Fundamental Differentiation Rules:**
  - State the constant, constant multiple, and power rules.
- **Apply Basic Combination Rules:**
  - Apply the sum and difference rules to combine derivatives.
- **Differentiate Exponential Functions:**
  - Compute the derivative of e^x.
- **Product and Quotient Rules:**
  - Use the product rule to find the derivative of a product of functions.
  - Use the quotient rule to find the derivative of a quotient of functions.
- **Extend Differentiation Techniques:**
  - Extend the power rule to functions with negative exponents.
- **Combine Rules for Complex Functions:**
  - Combine the differentiation rules to find the derivative of a polynomial or rational function.

The Constant Rule

Let c be a constant.
If $f(x) = c$, then $f'(x) = 0$.
Alternatively, we may express this rule as:

$$\frac{d}{dx}(c) = 0$$

The Power Rule

Let n be a positive integer. If $f(x) = x^n$, then:

$$f'(x) = nx^{n-1}$$

Alternatively, we may express this rule as:

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Sum and Difference Rules

Sum Rule

Let $f(x)$ and $g(x)$ be differentiable functions. The derivative of the sum is:

$$\frac{d}{dx}(f(x)+g(x)) = \frac{d}{dx}(f(x)) + \frac{d}{dx}(g(x))$$

That is, for $j(x) = f(x) + g(x)$:

$$j'(x) = f'(x) + g'(x)$$

Difference Rule

The derivative of the difference is:

$$\frac{d}{dx}(f(x)-g(x)) = \frac{d}{dx}(f(x)) - \frac{d}{dx}(g(x))$$

That is, for $j(x) = f(x) - g(x)$:

$$j'(x) = f'(x) - g'(x)$$

Constant Multiple Rule

Constant Multiple Rule

Let k be a constant. The derivative of a constant multiplied by a function is:

$$\frac{d}{dx}(kf(x)) = k\frac{d}{dx}(f(x))$$

That is, for $j(x) = kf(x)$:

$$j'(x) = kf'(x)$$

Product Rule

Product Rule

Let $f(x)$ and $g(x)$ be differentiable functions. Then:

$$\frac{d}{dx}(f(x) \cdot g(x)) = \frac{d}{dx}(f(x)) \cdot g(x) + \frac{d}{dx}(g(x)) \cdot f(x)$$

That is, if $j(x) = f(x) \cdot g(x)$:

$$j'(x) = f'(x) \cdot g(x) + g'(x) \cdot f(x)$$

This means that the derivative of a product of two functions is the derivative of the first function times the second function plus the derivative of the second function times the first function.

Exercise: Applying the Product Rule

Exercise: For $j(x) = (x^2 + 2)(3x^3 - 5x)$, find $j'(x)$ by applying the product rule.

Exercise 1: Applying the Product Rule

Exercise: For $h(x) = (4x^3 - x)(2x + 7)$, find $h'(x)$ using the product rule.

Confirm your result by first expanding the product and then differentiating.

Exercise 2: Applying the Quotient Rule

Exercise: Differentiate $m(x) = (x^2 - 5x + 6)(x - 3)^{-1}$ using the product rule and power rule.

Simplify the function by multiplying out and then differentiate to verify your result.

Exercise 3: Sum and Constant Multiple Rules

Exercise: If $p(x) = 7x^5 - 3x^2 + 4x - 9$, find $p'(x)$ by applying the sum and constant multiple rules.

Exercise 4: Combined Differentiation Rules

Exercise: For $f(x) = (5x^4 + 3x)(x^3 - 2x + 1)$, use the product rule to find $f'(x)$.

Check the result by expanding $f(x)$ and then differentiating.

The Quotient Rule

The Quotient Rule

Let $f(x)$ and $g(x)$ be differentiable functions. Then:

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x) \cdot g(x) - g'(x) \cdot f(x)}{(g(x))^2}$$

That is, if $j(x) = \frac{f(x)}{g(x)}$, then:

$$j'(x) = \frac{f'(x) \cdot g(x) - g'(x) \cdot f(x)}{(g(x))^2}$$

This rule states that the derivative of the quotient of two functions is the difference of the derivative of the numerator times the denominator and the derivative of the denominator times the numerator, all over the square of the denominator.

Exercise: Applying the Quotient Rule

Exercise 1: For $j(x) = \frac{3x^2+5}{x-4}$, find $j'(x)$ using the quotient rule. Simplify your answer as much as possible.

Exercise 2: Differentiate $m(x) = \frac{2x^3-x+1}{x^2+2}$ by applying the quotient rule. Verify your result by expanding and simplifying.

Exercise 3: For $h(x) = \frac{x^2+4x-1}{3x^2-5}$, use the quotient rule to find $h'(x)$ and express your answer in simplest form.

Extended Power Rule

Extended Power Rule

If k is a negative integer, then:

$$\frac{d}{dx}(x^k) = kx^{k-1}$$

This rule extends the power rule to include cases where the exponent is negative, allowing us to differentiate functions with negative exponents.

Exercise: Applying the Extended Power Rule

Exercise 1: Differentiate $f(x) = x^{-3}$ using the extended power rule. Simplify your answer.

Exercise 2: Find the derivative of $g(x) = \frac{1}{x^4}$ by rewriting the function with a negative exponent and then applying the extended power rule.

Exercise 3: For $h(x) = \frac{5}{x^2}$, express $h(x)$ as $5x^{-2}$ and then find $h'(x)$ using the extended power rule.

Determining Where a Function Has a Horizontal Tangent

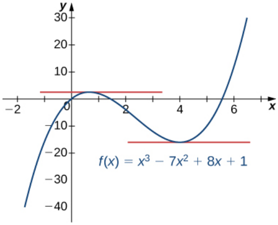
Problem: Determine the values of x for which $f(x) = x^3 - 7x^2 + 8x + 1$ has a horizontal tangent line.

Solution: To find the values of x for which $f(x)$ has a horizontal tangent line, we must solve $f'(x) = 0$.

Since:

$$f'(x) = 3x^2 - 14x + 8 = (3x - 2)(x - 4)$$

we need to solve: $(3x - 2)(x - 4) = 0$
Thus, the function has horizontal tangent lines at: $x = \frac{2}{3}$ and $x = 4$ as shown in the following graph.



Finding the Initial Velocity

Problem: The position of an object on a coordinate axis at time t is given by $s(t) = \frac{t}{t^2+1}$. What is the initial velocity of the object?

Finding the Initial Velocity

Problem: The position of an object on a coordinate axis at time t is given by $s(t) = \frac{t}{t^2+1}$. What is the initial velocity of the object?

Solution: The initial velocity is $v(0) = s'(0)$. To find $s'(t)$, we apply the quotient rule:

$$s'(t) = \frac{1 \cdot (t^2+1) - 2t \cdot t}{(t^2+1)^2} = \frac{1-t^2}{(t^2+1)^2}$$

Evaluating at $t = 0$:

$$v(0) = s'(0) = \frac{1-0^2}{(0^2+1)^2} = \frac{1}{1} = 1$$

Therefore, the initial velocity of the object is 1.

Derivative of the Natural Exponential Function

Derivative

Let $f(x) = e^x$ be the natural exponential function. Then:

$$f'(x) = e^x$$

Example: Find the derivative of $y = \frac{e^x}{x}$.
Solution: Use the derivative of the natural exponential function and the quotient rule:

$$y' = \frac{(e^x) \cdot x - 1 \cdot e^x}{x^2}$$

Applying the quotient rule and simplifying:

$$y' = \frac{e^x(x-1)}{x^2}$$

Therefore, the derivative of $y = \frac{e^x}{x}$ is $\frac{e^x(x-1)}{x^2}$.

Finding the Derivative Using the Product Rule

Problem: Find the derivative of $h(x) = (x^2 + 1)e^x$.

Hint: Don't forget to use the product rule.

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Problem: Find the derivative of $h(x) = (x^2 + 1)e^x$.

Hint: Don't forget to use the product rule.

Solution: Using the product rule, we have:

$$h'(x) = \frac{d}{dx}(x^2 + 1) \cdot e^x + (x^2 + 1) \cdot \frac{d}{dx}(e^x)$$

Simplify each term:

$$h'(x) = 2x \cdot e^x + (x^2 + 1) \cdot e^x$$

Therefore, the derivative of $h(x) = (x^2 + 1)e^x$ is:

$$h'(x) = \left(2x + x^2 + 1\right) e^x$$

Derivatives as Rates of Change (Velocity/ Acceleration only)

Clotilde Djuikem

Learning Objectives

- Determine a new value of a quantity from its old value and the amount of change.
- Calculate the average rate of change and distinguish it from the instantaneous rate of change.
- Apply rates of change to displacement, velocity, and acceleration for motion along a straight line.
- Predict future population from present value and population growth rate.
- Use derivatives to calculate marginal cost and revenue in business.

Amount of Change Formula

One application of derivatives is to estimate an unknown value using the known value and its rate of change:

$$f(a+h) \approx f(a) + f'(a)h$$

where $f(a)$ is the known value, $f'(a)$ is the rate of change, and h is the change in the interval.
Example: If $f(3) = 2$ and $f'(3) = 5$, estimate $f(3.2)$:

$$f(3.2) \approx 2 + 0.2(5) = 3$$

Derivatives as Rates of Change

Derivatives are used to represent rates of change in various contexts. In physics, they are particularly important in understanding:

- **Velocity:** The rate of change of position with respect to time.
- **Acceleration:** The rate of change of velocity with respect to time.

Definition

Velocity, Speed, and Acceleration

Let $s(t)$ be a function giving the position of an object at time t.

- The **velocity** of the object at time t is given by $v(t) = s'(t)$.
- The **speed** of the object at time t is given by $|v(t)|$.
- The **acceleration** of the object at time t is given by $a(t) = v'(t) = s''(t)$.

Instantaneous vs. Average Velocity

Instantaneous Velocity:

The velocity at a specific moment in time, given by the derivative of the position function.

$$v(t) = \lim_{h \to 0} \frac{s(t+h) - s(t)}{h}$$

Average Velocity:

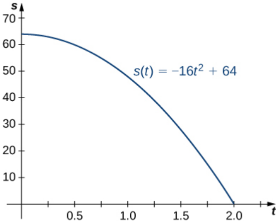
The total change in position divided by the total change in time over an interval $[a, b]$:

$$v_{\text{avg}} = \frac{s(b) - s(a)}{b - a}$$

Velocity Problem: Ball Dropped from 64 Feet

A ball is dropped from a height of 64 feet. Its height above ground (in feet) t seconds later is given by:

$$s(t) = -16t^2 + 64$$



- **a. Instantaneous Velocity:** What is the instantaneous velocity of the ball when it hits the ground?
- **b. Average Velocity:** What is the average velocity during its fall?

Solution: Instantaneous and Average Velocity

a. Instantaneous Velocity:

We first determine the time when the ball reaches the ground by solving $s(t) = 0$:

$$-16t^2 + 64 = 0 \quad \Rightarrow \quad t^2 = 4 \quad \Rightarrow \quad t = 2\,\text{seconds}$$

The velocity function is the derivative of the position function:

$$v(t) = s'(t) = \frac{d}{dt}(-16t^2 + 64) = -32t$$

Thus, the instantaneous velocity at $t = 2$ is:

$$v(2) = -32(2) = -64\,\text{ft/s}$$

b. Average Velocity:

The average velocity during the fall is given by:

$$v_{\text{avg}} = \frac{s(2) - s(0)}{2 - 0} = \frac{0 - 64}{2} = -32\,\text{ft/s}$$

Therefore, the average velocity is -32 ft/s, and the instantaneous velocity when the ball hits the ground is -64 ft/s.

Interpreting the Relationship between $v(t)$ and $a(t)$

A particle moves along a coordinate axis in the positive direction. Its position at time t is given by $s(t) = t^3 - 4t + 2$. Find $v(1)$ and $a(1)$, and use these values to answer the following questions:

1. Is the particle moving from left to right or from right to left at time $t = 1$?
2. Is the particle speeding up or slowing down at time $t = 1$?

Solution: Interpreting $v(t)$ and $a(t)$

Begin by finding $v(t)$ and $a(t)$:

$$v(t) = s'(t) = 3t^2 - 4 \quad \text{and} \quad a(t) = v'(t) = s''(t) = 6t$$

Evaluating these functions at $t = 1$:

$$v(1) = -1 \quad \text{and} \quad a(1) = 6$$

a. Is the particle moving from left to right or right to left? Since $v(1) < 0$, the particle is moving from right to left at $t = 1$.
b. Is the particle speeding up or slowing down? Since $v(1) < 0$ and $a(1) > 0$, velocity and acceleration are acting in opposite directions. This means the particle is **slowing down** because the acceleration is in the direction opposite to the velocity, causing $|v(t)|$ to decrease.

Position and Velocity

The position of a particle moving along a coordinate axis is given by:

$$s(t) = t^3 - 9t^2 + 24t + 4, \quad t \geq 0$$

1. Find $v(t)$.
2. At what time(s) is the particle at rest?
3. On what time intervals is the particle moving from left to right? From right to left?
4. Use the information obtained to sketch the path of the particle along a coordinate axis.

Solution: Velocity and Rest Points

a. Find $v(t)$: The velocity is the derivative of the position function:

$$v(t) = s'(t) = 3t^2 - 18t + 24$$

b. At what time(s) is the particle at rest? The particle is at rest when $v(t) = 0$. Solve $3t^2 - 18t + 24 = 0$. Factoring:

$$3(t-2)(t-4) = 0$$

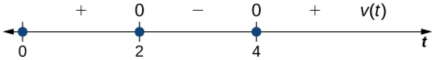
Thus, the particle is at rest at $t = 2$ and $t = 4$.

Solution: Motion Direction and Path

c. On what intervals is the particle moving left or right? The particle moves right when $v(t) > 0$ and left when $v(t) < 0$. Analyzing the sign of $v(t)$:

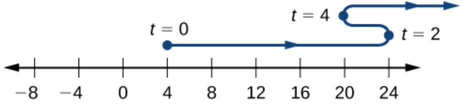
$$v(t) > 0 \quad \text{on} \quad [0, 2) \cup (4, \infty)$$

$$v(t) < 0 \quad \text{on} \quad (2, 4)$$



d. Sketch the path of the particle: The particle's position at $t = 0$ is $s(0) = 4$, at $t = 2$ is $s(2) = 24$, and at $t = 4$ is $s(4) = 20$. The particle changes direction at these points.

The sketch shows the particle starting at 4, moving to 24 at $t = 2$, then back to 20 at $t = 4$.



Motion Direction at $t = 3$

A particle moves along a coordinate axis. Its position at time t is given by:

$$s(t) = t^2 - 5t + 1$$

Is the particle moving from right to left or from left to right at time $t = 3$?
Hint: Find $v(3)$ and look at the sign of the velocity to determine the direction of motion.

Exercises: Velocity and Acceleration

For the following exercises, the given functions represent the position of a particle traveling along a horizontal line.

1. Find the velocity and acceleration functions.
2. Determine the time intervals when the object is slowing down or speeding up.

1. Position Function

$$s(t) = 2t^3 - 3t^2 - 12t + 8$$

2. Position Function

$$s(t) = 2t^3 - 15t^2 + 36t - 10$$

1. Find the velocity and acceleration functions for the given position function:

$$s(t) = \frac{t}{1 + t^2}$$

2. A rocket is fired vertically upward from the ground. The distance s in feet that the rocket travels from the ground after t seconds is given by:

$$s(t) = -16t^2 + 560t$$

- **a.** Find the velocity of the rocket 3 seconds after being fired.
- **b.** Find the acceleration of the rocket 3 seconds after being fired.

Summary: Velocity and Acceleration

- **Velocity** is the derivative of position, $v(t) = s'(t)$.
- **Acceleration** is the derivative of velocity, $a(t) = v'(t) = s''(t)$.
- Instantaneous velocity gives the rate of change at a specific moment, while average velocity gives the rate of change over a time interval.

Derivatives of Trigonometric Functions

Clotilde Djuikem

Learning Objectives

- Find the derivatives of the sine and cosine functions.
- Find the derivatives of the standard trigonometric functions.
- Calculate the higher-order derivatives of the sine and cosine functions.

The Derivatives of $\sin x$ and $\cos x$

Derivative of the Sine Function

$$\frac{d}{dx}(\sin x) = \cos x$$

Derivative of the Cosine Function

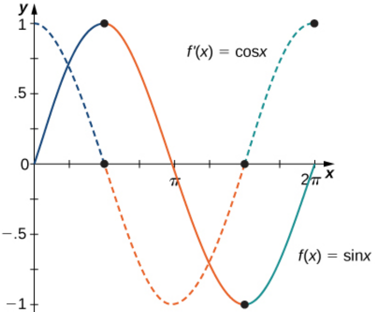
$$\frac{d}{dx}(\cos x) = -\sin x$$

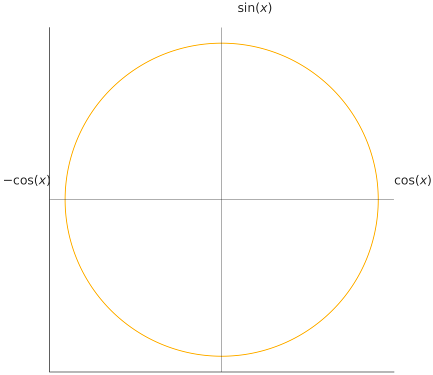
Proof

Derivative of a Function

The derivative of a function $f(x)$ at a point x is defined as:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$





Differentiating a Function Containing $\sin x$

Example 1: Find the derivative of $f(x) = 5x^3 \sin x$.

Example 2: Find the derivative of $f(x) = \sin x \cos x$.

Example 3: Find the derivative of $g(x) = \frac{\cos x}{4x^2}$.

Example 4: Find the derivative of $f(x) = \frac{x}{\cos x}$.

Particle Motion and Rest Time

A particle moves along a coordinate axis in such a way that its position at time t is given by:

$$s(t) = 2\sin t - t$$

for $0 \leq t \leq 2\pi$.

Question

At what times is the particle at rest?

Particle Motion and Rest Time

A particle moves along a coordinate axis. Its position at time t is given by:

$$s(t) = \sqrt{3}t + 2\cos t$$

for $0 \leq t \leq 2\pi$.

Question

At what times is the particle at rest?

The Derivative of the Tangent Function

Example: Find the derivative of $f(x) = \tan x$.

The Derivative of the Tangent Function

Example: Find the derivative of $f(x) = \tan x$.

Solution

To differentiate $f(x) = \tan x$, recall that $\tan x = \frac{\sin x}{\cos x}$ and use the quotient rule:

$$f'(x) = \frac{\cos x \cdot \frac{d}{dx}(\sin x) - \sin x \cdot \frac{d}{dx}(\cos x)}{\cos^2 x}$$

Substitute the derivatives:

$$f'(x) = \frac{\cos x \cdot \cos x + \sin x \cdot \sin x}{\cos^2 x}$$

Simplify the expression using the Pythagorean identity $\sin^2 x + \cos^2 x = 1$:

$$f'(x) = \frac{1}{\cos^2 x}$$

Therefore:

$$f'(x) = \sec^2 x$$

The Derivative of the Cotangent Function

Example: Find the derivative of $f(x) = \cot x$.

The Derivative of the Cotangent Function

Example: Find the derivative of $f(x) = \cot x$.

Solution

To differentiate $f(x) = \cot x$, recall that $\cot x = \frac{\cos x}{\sin x}$ and use the quotient rule:

$$f'(x) = \frac{\sin x \cdot \frac{d}{dx}(\cos x) - \cos x \cdot \frac{d}{dx}(\sin x)}{\sin^2 x}$$

Substitute the derivatives:

$$f'(x) = \frac{\sin x \cdot (-\sin x) - \cos x \cdot \cos x}{\sin^2 x}$$

Simplify the expression: $f'(x) = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x}$
Using the Pythagorean identity $\sin^2 x + \cos^2 x = 1$:

$$f'(x) = -\frac{1}{\sin^2 x} = f'(x) = -\csc^2 x$$

Derivatives of $\tan x$, $\cot x$, $\sec x$, and $\csc x$

Trigonometric Derivatives

The derivatives of the remaining trigonometric functions are as follows:

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

Equation of the Tangent Line

Find the equation of a line tangent to the graph of $f(x) = \cot x$ at $x = \frac{\pi}{4}$.

Formula for the Tangent Line

The equation of the tangent line at $x = x_0$ is given by:

$$y = f(x_0) + f'(x_0)(x - x_0)$$

Equation of the Tangent Line

Find the equation of a line tangent to the graph of $f(x) = \cot x$ at $x = \frac{\pi}{4}$.

Formula for the Tangent Line

The equation of the tangent line at $x = x_0$ is given by:

$$y = f(x_0) + f'(x_0)(x - x_0)$$

Steps

1. Calculate $f(x_0)$: Evaluate $f\left(\frac{\pi}{4}\right) = \cot\left(\frac{\pi}{4}\right)$.
2. Calculate $f'(x_0)$ this means **Find the slope**: Find the derivative $f'(x) = -\csc^2 x$ and evaluate $f'\left(\frac{\pi}{4}\right)$.
3. Apply the formula: Substitute $x_0 = \frac{\pi}{4}$, $f(x_0)$, and $f'(x_0)$ into the formula to get the equation of the tangent line.

Finding the Derivative of Trigonometric Functions

Example: Find the derivative of $f(x) = \csc x + x \tan x$.

Finding the Derivative of Trigonometric Functions

Example: Find the derivative of $f(x) = \csc x + x \tan x$.

Solution

To differentiate $f(x) = \csc x + x \tan x$, we need to apply the sum and product rules:

$$f'(x) = \frac{d}{dx}(\csc x) + \frac{d}{dx}(x \tan x)$$

Start by differentiating each term separately: $\frac{d}{dx}(\csc x) = -\csc x \cot x$ For the second term, use the product rule:

$$\frac{d}{dx}(x \tan x) = \frac{d}{dx}(x) \cdot \tan x + x \cdot \frac{d}{dx}(\tan x)$$

Substitute the derivatives: $= 1 \cdot \tan x + x \cdot \sec^2 x$
Putting it all together:

$$f'(x) = -\csc x \cot x + \tan x + x \sec^2 x$$

Example: Find the derivative of $f(x) = 2\tan x - 3\cot x$.

Solution

To differentiate $f(x) = 2\tan x - 3\cot x$, apply the constant multiple rule to each term:

$$f'(x) = 2\frac{d}{dx}(\tan x) - 3\frac{d}{dx}(\cot x)$$

Substitute the derivatives of $\tan x$ and $\cot x$:

$$f'(x) = 2\sec^2 x - 3(-\csc^2 x)$$

Simplify the expression:

$$f'(x) = 2\sec^2 x + 3\csc^2 x$$

Finding the Slope of the Tangent Line

Find the slope of the line tangent to the graph of $f(x) = \tan x$ at $x = \frac{\pi}{6}$.

Finding the Slope of the Tangent Line

Find the slope of the line tangent to the graph of $f(x) = \tan x$ at $x = \frac{\pi}{6}$.

Solution

First, find the derivative $f'(x)$:

$$f'(x) = \sec^2 x$$

Next, evaluate the derivative at $x = \frac{\pi}{6}$:

$$f'\left(\frac{\pi}{6}\right) = \sec^2\left(\frac{\pi}{6}\right)$$

Since $\sec\left(\frac{\pi}{6}\right) = \frac{2}{\sqrt{3}}$, we have:

$$f'\left(\frac{\pi}{6}\right) = \left(\frac{2}{\sqrt{3}}\right)^2 = \frac{4}{3}$$

Therefore, the slope of the tangent line at $x = \frac{\pi}{6}$ is $\frac{4}{3}$.

Finding Higher-Order Derivatives of $y = \sin x$

Example: Find the first four derivatives of $y = \sin x$.

Solution

Calculate each derivative in sequence:

1. **First derivative:**
$$y' = \frac{d}{dx}(\sin x) = \cos x$$
2. **Second derivative:**
$$y'' = \frac{d}{dx}(\cos x) = -\sin x$$
3. **Third derivative:**
$$y''' = \frac{d}{dx}(-\sin x) = -\cos x$$
4. **Fourth derivative:**
$$y^{(4)} = \frac{d}{dx}(-\cos x) = \sin x$$

The derivatives repeat in a cycle every four derivatives:
$\sin x \to \cos x \to -\sin x \to -\cos x \to \sin x$.

Analysis of Higher-Order Derivatives for $\sin x$

Once we recognize the pattern of derivatives, we can determine any higher-order derivative of $\sin x$ by identifying its position in the cycle.

Pattern Recognition

The derivatives of $\sin x$ follow a repeating cycle every four derivatives:

$$\sin x \to \cos x \to -\sin x \to -\cos x \to \sin x$$

Therefore:

- Every fourth derivative equals $\sin x$:

$$\frac{d^4}{dx^4}(\sin x) = \frac{d^8}{dx^8}(\sin x) = \frac{d^{12}}{dx^{12}}(\sin x) = \cdots = \frac{d^{4n}}{dx^{4n}}(\sin x) = \sin x$$

- Every $(4n+1)$ derivative equals $\cos x$:

$$\frac{d^5}{dx^5}(\sin x) = \frac{d^9}{dx^9}(\sin x) = \frac{d^{13}}{dx^{13}}(\sin x) = \cdots = \frac{d^{4n+1}}{dx^{4n+1}}(\sin x) = \cos x$$

Finding the Fourth Derivative of $y = \cos x$

Example: Find $\frac{d^4 y}{dx^4}$ for $y = \cos x$.

Finding the Fourth Derivative of $y = \cos x$

Example: Find $\frac{d^4y}{dx^4}$ for $y = \cos x$.

Solution

Calculate each derivative in sequence:

1. **First derivative:** $y' = -\sin x$
2. **Second derivative:** $y'' = -\cos x$
3. **Third derivative:** $y''' = \sin x$
4. **Fourth derivative:** $y^{(4)} = \cos x$

The derivatives of $\cos x$ follow a repeating cycle every four derivatives:
$\cos x \to -\sin x \to -\cos x \to \sin x \to \cos x$.
Therefore, $\frac{d^4y}{dx^4} = \cos x$.

Using the Pattern for Higher-Order Derivatives of $y = \sin x$

Example: Find $\frac{d^{74}}{dx^{74}}(\sin x)$.

Using the Pattern for Higher-Order Derivatives of $y = \sin x$

Example: Find $\frac{d^{74}}{dx^{74}}(\sin x)$.

Solution

The derivatives of $\sin x$ follow a repeating cycle every four derivatives:

$$\sin x \to \cos x \to -\sin x \to -\cos x \to \sin x$$

To find the 74th derivative, determine its position in the cycle by dividing 74 by 4:

$$74 \div 4 = 18 \quad \text{remainder } 2$$

A remainder of 2 corresponds to $-\sin x$.

$$\therefore \quad \frac{d^{74}}{dx^{74}}(\sin x) = -\sin x$$

Hint for Finding Higher-Order Derivatives of $y = \sin x$

To find $\frac{d^{59}}{dx^{59}}(\sin x)$, observe the cycle of derivatives which repeats every four derivatives:

$$\sin x \to \cos x \to -\sin x \to -\cos x \to \sin x$$

Use the formula:

$$\frac{d^{59}}{dx^{59}}(\sin x) = \frac{d^{4(14)+3}}{dx^{4(14)+3}}(\sin x)$$

The remainder when 59 is divided by 4 is 3, indicating the position in the cycle.

$$\frac{d^{4(14)+3}}{dx^{4(14)+3}}(\sin x) = -\cos x$$

Thus, by identifying the remainder, you can determine the corresponding function in the cycle.

Analyzing Velocity and Acceleration

A particle moves along a coordinate axis such that its position at time t is given by:

$$s(t) = 2 - \sin t$$

Step 1: Find the Velocity

The velocity is the first derivative of $s(t)$: $v(t) = s'(t) = -\cos t$ Evaluate $v\left(\frac{\pi}{4}\right)$: $v\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$

Step 2: Find the Acceleration

The acceleration is the derivative of $v(t)$: $a(t) = v'(t) = \sin t$
Evaluate $a\left(\frac{\pi}{4}\right)$: $a\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

Conclusion

Since $v\left(\frac{\pi}{4}\right) < 0$ and $a\left(\frac{\pi}{4}\right) > 0$, velocity and acceleration are in opposite directions. This indicates the particle is slowing down.

Analyzing Velocity and Acceleration of a Spring-Mass System

A block attached to a spring is moving vertically with position given by:

$$s(t) = 2\sin t$$

Step 1: Find the Velocity

The velocity is the first derivative of $s(t)$: $v(t) = s'(t) = 2\cos t$ Evaluate $v\left(\frac{5\pi}{6}\right)$: $v\left(\frac{5\pi}{6}\right) = -\sqrt{3}$

Step 2: Find the Acceleration

The acceleration is the derivative of $v(t)$: $a(t) = v'(t) = -2\sin t$ Evaluate $a\left(\frac{5\pi}{6}\right)$: $a\left(\frac{5\pi}{6}\right) = -1$

Conclusion

Both $v\left(\frac{5\pi}{6}\right)$ and $a\left(\frac{5\pi}{6}\right)$ are negative, indicating that velocity and acceleration are in the same direction. Therefore, the block is speeding up.

Key Concepts and Equations

Key Concepts

We can find the derivatives of $\sin x$ and $\cos x$ by using the definition of the derivative and the limit formulas:

$$\frac{d}{dx}(\sin x) = \cos x \quad \text{and} \quad \frac{d}{dx}(\cos x) = -\sin x$$

With these formulas, we can determine the derivatives of all six basic trigonometric functions.

Key Equations

- Derivative of sine function:
$$\frac{d}{dx}(\sin x) = \cos x$$
- Derivative of cosine function:
$$\frac{d}{dx}(\cos x) = -\sin x$$
- Derivative of tangent function:
$$\frac{d}{dx}(\tan x) = \sec^2 x$$
- Derivative of cotangent function:
$$\frac{d}{dx}(\cot x) = -\csc^2 x$$
- Derivative of secant function:
$$\frac{d}{dx}(\sec x) = \sec x \tan x$$
- Derivative of cosecant function:
$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

The Chain Rule

Clotilde Djuikem

Learning Objectives

By the end of this lesson, you should be able to:

- State the chain rule for the composition of two functions.
- Apply the chain rule together with the power rule.
- Correctly apply the chain rule with the product or quotient rules when both are necessary.
- Accurately apply the chain rule with exponential and trigonometric functions when both are necessary.
- Recognize the chain rule for compositions involving three or more functions.
- Describe the proof of the chain rule.

Rule: The Chain Rule

Let f and g be functions. For all x in the domain of g where g is differentiable at x and f is differentiable at $g(x)$, the derivative of the composite function

$$h(x) = (f \circ g)(x) = f(g(x))$$

is given by:

$$h'(x) = f'(g(x)) \cdot g'(x)$$

Alternatively, if y is a function of u, and u is a function of x, then:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Problem-Solving Strategy: Applying the Chain Rule

To differentiate $h(x) = f(g(x))$, follow these steps:

1. **Identify** $f(x)$ and $g(x)$.
2. **Differentiate** $f(x)$: Find $f'(x)$ and evaluate it at $g(x)$ to obtain $f'(g(x))$.
3. **Differentiate** $g(x)$: Find $g'(x)$.
4. **Apply the Chain Rule**: Write

$$h'(x) = f'(g(x)) \cdot g'(x)$$

Note

When applying the chain rule to the composition of two or more functions, start from the outside function and work inwards. The derivative of a composition with two functions has two parts, three functions have three parts, and so on. Remember that you should never evaluate a derivative at another derivative.

Rule: Power Rule for Composition of Functions

For all values of x where the derivative is defined, if

$$h(x) = (g(x))^n,$$

then the derivative is given by:

$$h'(x) = n \cdot (g(x))^{n-1} \cdot g'(x).$$

This rule is a combination of the power rule and the chain rule, allowing us to differentiate compositions where the outer function is a power.

Using the Chain and Power Rules

Find the derivative of $h(x) = \frac{1}{(3x^2+1)^2}$.

Using the Chain and Power Rules

Find the derivative of $h(x) = \frac{1}{(3x^2+1)^2}$.

Solution

First, rewrite $h(x)$ using negative exponents:

$$h(x) = (3x^2 + 1)^{-2}$$

Let $g(x) = 3x^2 + 1$. Then $h(x) = (g(x))^{-2}$.
Applying the power rule, we have: $h'(x) = -2(g(x))^{-3} \cdot g'(x)$
Calculate $g'(x)$: $g'(x) = 6x$
Substitute $g(x)$ and $g'(x)$: $h'(x) = -2(3x^2 + 1)^{-3} \cdot 6x$ Simplify:

$$h'(x) = -12x(3x^2 + 1)^{-3}$$

Rewriting back to the original form gives:

$$h'(x) = \frac{-12x}{(3x^2 + 1)^3}$$

Using the Chain and Power Rules

Find the derivative of $h(x) = (2x^3 + 2x - 1)^4$.

Using the Chain and Power Rules

Find the derivative of $h(x) = (2x^3 + 2x - 1)^4$.

Solution

Let $g(x) = 2x^3 + 2x - 1$. Then $h(x) = (g(x))^4$.
Applying the power rule with the chain rule, we have:

$$h'(x) = 4(g(x))^3 \cdot g'(x)$$

Calculate $g'(x)$:

$$g'(x) = 6x^2 + 2$$

Substitute $g(x)$ and $g'(x)$ into the derivative:

$$h'(x) = 4(2x^3 + 2x - 1)^3 \cdot (6x^2 + 2)$$

Simplify the expression:

$$h'(x) = (24x^2 + 8)(2x^3 + 2x - 1)^3$$

Using the Chain and Power Rules with a Trigonometric Function

Find the derivative of $h(x) = \sin^3(x)$.

Using the Chain and Power Rules with a Trigonometric Function

Find the derivative of $h(x) = \sin^3(x)$.

Solution

Rewrite $h(x)$ as $h(x) = (\sin(x))^3$.
Let $g(x) = \sin(x)$. Then $h(x) = (g(x))^3$.
Applying the power rule with the chain rule, we have:

$$h'(x) = 3(\sin(x))^2 \cdot \cos(x)$$

The derivative $h'(x)$ is therefore:

$$h'(x) = 3\sin^2(x) \cdot \cos(x)$$

This expression simplifies to:

$$h'(x) = 3\sin^2(x)\cos(x)$$

Finding the Tangent Line Equation

Find the equation of the line tangent to the graph of $h(x) = \frac{1}{(3x-5)^2}$ at $x = 2$.

Finding the Tangent Line Equation

Find the equation of the line tangent to the graph of $h(x) = \frac{1}{(3x-5)^2}$ at $x = 2$.

Solution

1. Rewrite the Function:

$$h(x) = (3x-5)^{-2}$$

2. Find the Derivative $h'(x)$: Using the power and chain rules:
$h'(x) = -2(3x-5)^{-3} \cdot 3$ Simplify: $h'(x) = -\frac{6}{(3x-5)^3}$
3. Evaluate $h(x)$ and $h'(x)$ at $x = 2$: $h(2) = \frac{1}{(3(2)-5)^2} = \frac{1}{1} = 1$

$$h'(2) = -\frac{6}{(3(2)-5)^3} = -6$$

4. Use the Point-Slope Formula: The equation of the tangent line at $x = 2$ is given by: $y - 1 = -6(x-2)$ Simplify:

$$y = -6x + 13$$

The equation of the tangent line is: $y = -6x + 13$

Finding the Tangent Line Equation

Find the equation of the line tangent to the graph of $f(x) = (x^2 - 2)^3$ at $x = -2$.

Finding the Tangent Line Equation

Find the equation of the line tangent to the graph of $f(x) = (x^2 - 2)^3$ at $x = -2$.

Solution

1. Find the Derivative $f'(x)$: Using the chain rule and power rule:

$$f'(x) = 3(x^2 - 2)^2 \cdot (2x) = 6x(x^2 - 2)^2$$

2. Evaluate $f(x)$ and $f'(x)$ at $x = -2$:

$$f(-2) = ((-2)^2 - 2)^3 = (4 - 2)^3 = 2^3 = 8$$

$$f'(-2) = 6(-2)((-2)^2 - 2)^2 = 6(-2)(4 - 2)^2 = 6(-2)(2)^2 = -48$$

3. Use the Point-Slope Formula: The equation of the tangent line at $x = -2$ is given by:

$$y - 8 = -48(x + 2)$$

Simplify: $y = -48x - 88$

Using the Chain Rule on a General Cosine Function

Find the derivative of $h(x) = \cos(g(x))$.

Using the Chain Rule on a General Cosine Function

Find the derivative of $h(x) = \cos(g(x))$.

Solution

Think of $h(x) = \cos(g(x))$ as $f(g(x))$, where $f(x) = \cos x$. Since $f'(x) = -\sin x$, we have:

$$f'(g(x)) = -\sin(g(x))$$

Applying the Chain Rule:

$$h'(x) = f'(g(x)) \cdot g'(x)$$

Substitute $f'(g(x)) = -\sin(g(x))$:

$$h'(x) = -\sin(g(x)) \cdot g'(x)$$

Thus, the derivative of $h(x) = \cos(g(x))$ is given by:

$$h'(x) = -\sin(g(x)) \cdot g'(x)$$

Using the Chain Rule on a Cosine Function

Find the derivative of $h(x) = \cos(5x^2)$.

Using the Chain Rule on a Cosine Function

Find the derivative of $h(x) = \cos(5x^2)$.

Solution

Let $g(x) = 5x^2$, then $g'(x) = 10x$.
Since $h(x) = \cos(g(x))$, we use the result from the previous example:

$$h'(x) = -\sin(g(x)) \cdot g'(x)$$

Substitute $g(x) = 5x^2$ and $g'(x) = 10x$:

$$h'(x) = -\sin(5x^2) \cdot 10x$$

Simplify the expression:

$$h'(x) = -10x \sin(5x^2)$$

Using the Chain Rule on Another Trigonometric Function

Find the derivative of $h(x) = \sec(4x^5 + 2x)$.

Using the Chain Rule on Another Trigonometric Function

Find the derivative of $h(x) = \sec(4x^5 + 2x)$.

Solution

Apply the chain rule to $h(x) = \sec(g(x))$, where:

$$h'(x) = \sec(g(x)) \cdot \tan(g(x)) \cdot g'(x)$$

In this problem, $g(x) = 4x^5 + 2x$, so:

$$g'(x) = 20x^4 + 2$$

Substitute $g(x)$ and $g'(x)$:

$$h'(x) = \sec(4x^5 + 2x) \cdot \tan(4x^5 + 2x) \cdot (20x^4 + 2)$$

Simplify:

$$h'(x) = (20x^4 + 2) \cdot \sec(4x^5 + 2x) \cdot \tan(4x^5 + 2x)$$

Using the Chain Rule on a Sine Function

Find the derivative of $h(x) = \sin(7x + 2)$.

Using the Chain Rule on a Sine Function

Find the derivative of $h(x) = \sin(7x + 2)$.

Solution

Let $g(x) = 7x + 2$. Then $h(x) = \sin(g(x))$.
Apply the Chain Rule:

$$h'(x) = \cos(g(x)) \cdot g'(x)$$

First, find $g'(x)$:

$$g'(x) = 7$$

Substitute $g(x) = 7x + 2$ and $g'(x) = 7$ into the derivative:

$$h'(x) = \cos(7x + 2) \cdot 7$$

Simplify the expression:

$$h'(x) = 7\cos(7x + 2)$$

Using the Chain Rule with Trigonometric Functions

For all values of x where the derivative is defined:

$$\frac{d}{dx}(\sin(g(x))) = \cos(g(x)) \cdot g'(x) \quad \text{or} \quad \frac{d}{dx}(\sin u) = \cos u \cdot \frac{du}{dx}$$

$$\frac{d}{dx}(\cos(g(x))) = -\sin(g(x)) \cdot g'(x) \quad \text{or} \quad \frac{d}{dx}(\cos u) = -\sin u \cdot \frac{du}{dx}$$

$$\frac{d}{dx}(\tan(g(x))) = \sec^2(g(x)) \cdot g'(x) \quad \text{or} \quad \frac{d}{dx}(\tan u) = \sec^2 u \cdot \frac{du}{dx}$$

$$\frac{d}{dx}(\cot(g(x))) = -\csc^2(g(x)) \cdot g'(x) \quad \text{or} \quad \frac{d}{dx}(\cot u) = -\csc^2 u \cdot \frac{du}{dx}$$

$$\frac{d}{dx}(\sec(g(x))) = \sec(g(x)) \tan(g(x)) \cdot g'(x) \quad \text{or} \quad \frac{d}{dx}(\sec u) = \sec u \tan u \cdot \frac{du}{dx}$$

$$\frac{d}{dx}(\csc(g(x))) = -\csc(g(x)) \cot(g(x)) \cdot g'(x) \quad \text{or} \quad \frac{d}{dx}(\csc u) = -\csc u \cot u$$

Finding the Derivative Using the Product and Chain Rules

Find the derivative of $h(x) = (2x + 1)^5(3x - 2)^7$.

Finding the Derivative Using the Product and Chain Rules

Find the derivative of $h(x) = (2x+1)^5(3x-2)^7$.

Solution

Since $h(x)$ is a product of two functions, $u(x) = (2x+1)^5$ and $v(x) = (3x-2)^7$, we apply the product rule: $h'(x) = u'(x)v(x) + u(x)v'(x)$ 1. Differentiate $u(x) = (2x+1)^5$ using the chain rule:

$$u'(x) = 5(2x+1)^4 \cdot 2 = 10(2x+1)^4$$

2. Differentiate $v(x) = (3x-2)^7$ using the chain rule:

$$v'(x) = 7(3x-2)^6 \cdot 3 = 21(3x-2)^6$$

3. Substitute into the product rule:

$$h'(x) = \left(10(2x+1)^4\right)(3x-2)^7 + (2x+1)^5\left(21(3x-2)^6\right)$$

4. Simplify the expression:

$$h'(x) = 10(2x+1)^4(3x-2)^7 + 21(2x+1)^5(3x-2)^6$$

Finding the Derivative Using the Product and Chain Rules

Find the derivative of $h(x) = x(2x+3)^3$.

Finding the Derivative Using the Product and Chain Rules

Find the derivative of $h(x) = x(2x+3)^3$.

Solution

Since $h(x)$ is a product of two functions, $u(x) = x$ and $v(x) = (2x+3)^3$, we apply the product rule: $h'(x) = u'(x)v(x) + u(x)v'(x)$

1. Differentiate $u(x) = x$:

$$u'(x) = 1$$

2. Differentiate $v(x) = (2x+3)^3$ using the chain rule:

$$v'(x) = 3(2x+3)^2 \cdot 2 = 6(2x+3)^2$$

3. Substitute into the product rule: $h'(x) = (1)(2x+3)^3 + x(6(2x+3)^2)$
4. Simplify the expression:

$$h'(x) = (2x+3)^3 + 6x(2x+3)^2$$

Using the Chain Rule with Exponential Functions

Chain Rule with Exponential

In general, for an exponential function $f(x) = e^{g(x)}$, the derivative is:

$$\frac{d}{dx}(e^{g(x)}) = e^{g(x)} \cdot g'(x)$$

Example: Find the derivative of $f(x) = e^{\tan(2x)}$.

Solution

Let $g(x) = \tan(2x)$. Then $f(x) = e^{g(x)}$.
Applying the chain rule:

$$f'(x) = e^{\tan(2x)} \cdot \frac{d}{dx}(\tan(2x))$$

Now, we need to differentiate $\tan(2x)$ and substitute into the expression for $f'(x)$.

Using the Chain Rule with Exponential Functions

Example: Find the derivative of $f(x) = e^{\tan(2x)}$.

Solution

Let $g(x) = \tan(2x)$. Then $f(x) = e^{g(x)}$.
Applying the chain rule:

$$f'(x) = e^{\tan(2x)} \cdot \frac{d}{dx}(\tan(2x))$$

Differentiate $\tan(2x)$: $\frac{d}{dx}(\tan(2x)) = \sec^2(2x) \cdot 2 = 2\sec^2(2x)$ Substitute back into the expression for $f'(x)$:

$$f'(x) = e^{\tan(2x)} \cdot 2\sec^2(2x)$$

Final solution:

$$f'(x) = 2\sec^2(2x) \cdot e^{\tan(2x)}$$

Finding the Derivative Using the Product and Chain Rules

Find the derivative of $h(x) = xe^{2x}$.

Solution

Since $h(x) = xe^{2x}$, we apply the product rule, where $u(x) = x$ and $v(x) = e^{2x}$:

$$h'(x) = u'(x)v(x) + u(x)v'(x)$$

1. Differentiate $u(x) = x$:

$$u'(x) = 1$$

2. Differentiate $v(x) = e^{2x}$ using the chain rule:

$$v'(x) = e^{2x} \cdot 2 = 2e^{2x}$$

3. Substitute into the product rule: $h'(x) = (1)e^{2x} + x(2e^{2x})$
4. Simplify the expression:

$$h'(x) = e^{2x}(1 + 2x)$$

Rule: Chain Rule for a Composition of Three Functions

For all values of x where the function is differentiable, if

$$k(x) = h(f(g(x))),$$

then the derivative $k'(x)$ is given by:

$$k'(x) = h'\big(f(g(x))\big) \cdot f'\big(g(x)\big) \cdot g'(x).$$

In other words, we are applying the chain rule twice:

1. First, find the derivative of the outermost function h with respect to $f(g(x))$.
2. Then, differentiate f with respect to $g(x)$.
3. Finally, differentiate g with respect to x.

This method allows us to handle compositions of three functions by working from the outermost function to the innermost.

Differentiating a Composite of Three Functions

Find the derivative of $k(x) = \cos^4(7x^2 + 1)$.

Solution

Rewrite $k(x)$ as $k(x) = (\cos(7x^2 + 1))^4$.
Let:

$$u = \cos(7x^2 + 1), \quad v = 7x^2 + 1$$

Then $k(x) = u^4$, and we will apply the Chain Rule in steps:
1. Differentiate $k(x) = u^4$ with respect to u: $\frac{d}{du}(u^4) = 4u^3$
2. Differentiate $u = \cos(v)$ with respect to v: $\frac{d}{dv}(\cos(v)) = -\sin(v)$
3. Differentiate $v = 7x^2 + 1$ with respect to x:

$$\frac{d}{dx}(7x^2 + 1) = 14x$$

Now, substitute back: $k'(x) = 4\cos^3(7x^2 + 1) \cdot (-\sin(7x^2 + 1)) \cdot 14x$
Simplify the expression:

$$k'(x) = -56x\cos^3(7x^2 + 1)\sin(7x^2 + 1)$$

Using the Chain Rule in a Velocity Problem

The position of a particle is given by:

$$s(t) = \sin(2t) + \cos(3t)$$

To find the velocity at $t = \frac{\pi}{6}$, first differentiate $s(t)$:

$$v(t) = 2\cos(2t) - 3\sin(3t)$$

Now, evaluate at $t = \frac{\pi}{6}$:

$$v\left(\frac{\pi}{6}\right) = 2\cos\left(\frac{\pi}{3}\right) - 3\sin\left(\frac{\pi}{2}\right)$$

Substitute values:

$$v\left(\frac{\pi}{6}\right) = 2 \cdot \frac{1}{2} - 3 \cdot 1 = -2$$

Therefore, $v\left(\frac{\pi}{6}\right) = -2$.

Finding the Acceleration of a Particle

The position of a particle is given by:

$$s(t) = \sin(4t)$$

1. First Derivative (Velocity):

$$v(t) = s'(t) = 4\cos(4t)$$

2. Second Derivative (Acceleration):

$$a(t) = v'(t) = -16\sin(4t)$$

Therefore, the acceleration at time t is:

$$a(t) = -16\sin(4t)$$

Proof of the Chain Rule (Informal)

To prove the Chain Rule for $h(x) = f(g(x))$, we start with the limit definition:

$$h'(a) = \lim_{x \to a} \frac{f(g(x)) - f(g(a))}{x - a}$$

Rewrite this expression:

$$h'(a) = \lim_{x \to a} \left(\frac{f(g(x)) - f(g(a))}{g(x) - g(a)} \cdot \frac{g(x) - g(a)}{x - a} \right)$$

Since $\lim_{x \to a} \frac{g(x)-g(a)}{x-a} = g'(a)$, we need to show that:

$$\lim_{x \to a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} = f'(g(a))$$

Substitute $y = g(x)$ and $b = g(a)$:

$$\lim_{x \to a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} = \lim_{y \to b} \frac{f(y) - f(b)}{y - b} = f'(b) = f'(g(a))$$

Therefore, we conclude: $h'(a) = f'(g(a)) \cdot g'(a)$

Using the Chain Rule with Functional Values

Given $h(x) = f(g(x))$ and the values:

$$g(1) = 4, \quad g'(1) = 3, \quad f'(4) = 7$$

Find $h'(1)$.

Solution

Using the Chain Rule:

$$h'(x) = f'(g(x)) \cdot g'(x)$$

Substitute $x = 1$:

$$h'(1) = f'(g(1)) \cdot g'(1)$$

Substitute the known values:

$$h'(1) = f'(4) \cdot 3 = 7 \cdot 3 = 21$$

Therefore, $h'(1) = 21$.

Using the Chain Rule with Functional Values

Given $h(x) = f(g(x))$ and the values:

$$g(2) = -3, \quad g'(2) = 4, \quad f'(-3) = 7$$

Find $h'(2)$.

Solution

Using the Chain Rule:

$$h'(x) = f'(g(x)) \cdot g'(x)$$

Substitute $x = 2$:

$$h'(2) = f'(g(2)) \cdot g'(2)$$

Substitute the known values:

$$h'(2) = f'(-3) \cdot 4 = 7 \cdot 4 = 28$$

Therefore, $h'(2) = 28$.

Rule: Chain Rule Using Leibniz's Notation

If y is a function of u, and u is a function of x, then the Chain Rule can be written as:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

This notation emphasizes that the derivative $\frac{dy}{dx}$ is the product of the derivative of y with respect to u and the derivative of u with respect to x.

Explanation

- Differentiate y with respect to u, which gives $\frac{dy}{du}$.
- Then, differentiate u with respect to x, giving $\frac{du}{dx}$.
- Multiply these results to find the derivative $\frac{dy}{dx}$.

Taking a Derivative Using Leibniz's Notation, Example 1

Find the derivative of $y = \left(\frac{x^3}{x+2}\right)^5$.

Solution

Let $u = \frac{x^3}{x+2}$, so $y = u^5$.

1. Apply the Chain Rule: $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$
2. Differentiate $y = u^5$ with respect to u: $\frac{dy}{du} = 5u^4$
3. Differentiate $u = \frac{x^3}{x+2}$ using the Quotient Rule:

$$\frac{du}{dx} = \frac{(x+2)(3x^2) - x^3(1)}{(x+2)^2} = \frac{3x^3 + 6x^2 - x^3}{(x+2)^2} = \frac{2x^3 + 6x^2}{(x+2)^2}$$

4. Combine and substitute u back:

$$\frac{dy}{dx} = 5\left(\frac{x^3}{x+2}\right)^4 \cdot \frac{2x^3 + 6x^2}{(x+2)^2}$$

Taking a Derivative Using Leibniz's Notation, Example 2

Find the derivative of $y = \tan(4x^2 - 3x + 1)$.

Taking a Derivative Using Leibniz's Notation, Example 2

Find the derivative of $y = \tan(4x^2 - 3x + 1)$.

Solution

Let $u = 4x^2 - 3x + 1$, so $y = \tan(u)$.

1. Apply the Chain Rule:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

2. Differentiate $y = \tan(u)$ with respect to u:

$$\frac{dy}{du} = \sec^2(u)$$

3. Differentiate $u = 4x^2 - 3x + 1$ with respect to x:

$$\frac{du}{dx} = 8x - 3$$

4. Combine the results: $\frac{dy}{dx} = \sec^2(4x^2 - 3x + 1) \cdot (8x - 3)$

Using Leibniz's Notation, Example

Find the derivative of $y = \cos(x^3)$, and express the answer entirely in terms of x.

Using Leibniz's Notation, Example

Find the derivative of $y = \cos(x^3)$, and express the answer entirely in terms of x.

Solution

Let $u = x^3$, so $y = \cos(u)$.
1. Apply the Chain Rule: $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$
2. Differentiate $y = \cos(u)$ with respect to u:

$$\frac{dy}{du} = -\sin(u)$$

3. Differentiate $u = x^3$ with respect to x: $\frac{du}{dx} = 3x^2$ 4. Combine the results and substitute $u = x^3$:

$$\frac{dy}{dx} = -\sin(x^3) \cdot 3x^2 = -3x^2 \sin(x^3)$$

Therefore, the derivative is: $\frac{dy}{dx} = -3x^2 \sin(x^3)$

Key Concepts: The Chain Rule

The chain rule allows us to differentiate compositions of two or more functions. For $h(x) = f(g(x))$:

$$h'(x) = f'(g(x)) \cdot g'(x)$$

In Leibniz's notation, the chain rule is expressed as:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Applications: - The chain rule can be used in combination with other differentiation rules. - For $h(x) = (g(x))^n$, the chain rule with the power rule becomes:

$$h'(x) = n(g(x))^{n-1} \cdot g'(x)$$

- For the composition of three functions $h(x) = f(g(k(x)))$, the rule extends to:

$$h'(x) = f'(g(k(x))) \cdot g'(k(x)) \cdot k'(x)$$

These formulations allow us to handle more complex compositions by breaking down the differentiation process step-by-step.

Implicit Differentiation

Clotilde Djuikem

Learning Objectives

- Find the derivative of a complicated function by using **implicit differentiation**.
- Use **implicit differentiation** to determine the equation of a tangent line.

Context: Why Implicit Differentiation?

- Not all functions can be written explicitly as $y = f(x)$. Some functions are defined **implicitly**, meaning the relationship between x and y is given by an equation involving both variables.
- For example, equations like $x^2 + y^2 = 1$ (the equation of a circle) define y implicitly in terms of x.
- In such cases, we need **implicit differentiation** to find derivatives, because it's difficult or impossible to solve for y explicitly.

Key Insight: Implicit Differentiation

Implicit differentiation allows us to differentiate both sides of an equation involving x and y, using the chain rule where necessary. This technique is crucial for dealing with equations where solving for y is not straightforward.

- We will explore a strategy to systematically apply implicit differentiation to such equations.

Problem-Solving Strategy: Implicit Differentiation

To perform **implicit differentiation** on an equation that defines a function y implicitly in terms of a variable x, follow these steps:

1. Take the derivative of both sides of the equation. Keep in mind that y is a function of x. Consequently:

$$\frac{d}{dx}(\sin x) = \cos x \quad \text{but} \quad \frac{d}{dx}(\sin y) = \cos y \cdot \frac{dy}{dx}$$

   because we must use the Chain Rule to differentiate $\sin y$ with respect to x.

2. Rewrite the equation so that all terms containing $\frac{dy}{dx}$ are on the left and all terms without $\frac{dy}{dx}$ are on the right.
3. Factor out $\frac{dy}{dx}$ on the left-hand side.
4. Solve for $\frac{dy}{dx}$ by dividing both sides of the equation by an appropriate algebraic expression.

Example: Using Implicit Differentiation

Problem:
Assume that y is defined implicitly by the equation $x^2 + y^2 = 25$. Find $\frac{dy}{dx}$.

Example: Using Implicit Differentiation

Problem:
Assume that y is defined implicitly by the equation $x^2 + y^2 = 25$. Find $\frac{dy}{dx}$.
Solution: Follow the steps in the problem-solving strategy.

1. **Differentiate both sides of the equation.**

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(25)$$

   Use the sum rule on the left side: $\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 0$

   Take the derivatives: $2x + 2y\frac{dy}{dx} = 0$

2. **Step 2:** Keep the terms with $\frac{dy}{dx}$ on the left and move the other terms to the right:

$$2y\frac{dy}{dx} = -2x$$

3. **Step 4:** Divide both sides of the equation by $2y$: $\frac{dy}{dx} = \frac{-x}{y}$ (Step 3 does not apply in this case.)

Thus, $\frac{dy}{dx} = \frac{-x}{y}$.

Analysis: Implicit Differentiation Result

Point 1

Note that the resulting expression for $\frac{dy}{dx}$ is in terms of both the independent variable x and the dependent variable y.

Point 2

Although in some cases it may be possible to express $\frac{dy}{dx}$ in terms of x only, it is generally not possible to do so.

Point 3

This highlights one of the key aspects of implicit differentiation: we work with both variables rather than isolating one entirely.

Example: ID and the Product Rule (Part 1)

Problem:

Assume that y is defined implicitly by the equation $x^3 \sin y + y = 4x + 3$. Find $\frac{dy}{dx}$.

Example: ID and the Product Rule (Part 1)

Problem:
Assume that y is defined implicitly by the equation $x^3 \sin y + y = 4x + 3$. Find $\frac{dy}{dx}$.
Solution:

1. **Step 1: Differentiate both sides of the equation.**

$$\frac{d}{dx}(x^3 \sin y + y) = \frac{d}{dx}(4x + 3)$$

ep 1.1: Apply the product rule to $x^3 \sin y$:

$$\frac{d}{dx}(x^3 \sin y) = 3x^2 \sin y + x^3 \cos y \frac{dy}{dx}$$

ep 1.2: Differentiate y with respect to x:

$$\frac{d}{dx}(y) = \frac{dy}{dx}$$

ep 1.3: Differentiate the right-hand side:

$$\frac{d}{dx}(4x + 3) = 4$$

Example: Using ID and the Product Rule (Part 2)

1. **Step 2:** Combine all derivatives:

$$3x^2 \sin y + x^3 \cos y \frac{dy}{dx} + \frac{dy}{dx} = 4$$

2. **Step 3:** Collect terms involving $\frac{dy}{dx}$:

$$\left(x^3 \cos y + 1\right) \frac{dy}{dx} = 4 - 3x^2 \sin y$$

3. **Step 4:** Solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{4 - 3x^2 \sin y}{x^3 \cos y + 1}$$

Thus, the derivative is $\frac{dy}{dx} = \frac{4-3x^2 \sin y}{x^3 \cos y+1}$.

Using ID to Find a Second Derivative (Part 1)

Problem: Find $\frac{d^2y}{dx^2}$ if $x^2 + y^2 = 25$.

Using ID to Find a Second Derivative (Part 1)

Problem: Find $\frac{d^2y}{dx^2}$ if $x^2 + y^2 = 25$.

Solution:

- **Step 1:** Recall from the implicit differentiation of the equation $x^2 + y^2 = 25$, we already know:

$$\frac{dy}{dx} = \frac{-x}{y}$$

- **Step 2:** Differentiate both sides of $\frac{dy}{dx} = \frac{-x}{y}$ with respect to x to find the second derivative $\frac{d^2y}{dx^2}$:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{-x}{y}\right)$$

- **Step 3:** Use the quotient rule to differentiate $\frac{-x}{y}$:

$$\frac{d}{dx}\left(\frac{-x}{y}\right) = \frac{-\left(1 \cdot y - x \cdot \frac{dy}{dx}\right)}{y^2}$$

Using ID to Find a Second Derivative (Part 2)

Continuing the Solution:

- **Step 4:** Substitute $\frac{dy}{dx} = \frac{-x}{y}$ into the equation:

$$\frac{d^2y}{dx^2} = \frac{-\left(y - x \cdot \frac{-x}{y}\right)}{y^2}$$

- **Step 5:** Simplify the expression:

$$\frac{d^2y}{dx^2} = \frac{-\left(y + \frac{x^2}{y}\right)}{y^2} = -\frac{y^2 + x^2}{y^3}$$

- **Step 6:** Since $x^2 + y^2 = 25$, substitute this into the numerator:

$$\frac{d^2y}{dx^2} = \frac{-25}{y^3}$$

Thus, the second derivative is $\frac{d^2y}{dx^2} = \frac{-25}{y^3}$.

Example: Using Implicit Differentiation

Problem:
Find $\frac{dy}{dx}$ for y defined implicitly by the equation:

$$4x^5 + \tan y = y^2 + 5x$$

Hint: Follow the problem-solving strategy, remembering to apply the chain rule to differentiate $\tan y$ and y^2.

Example: Using Implicit Differentiation

Problem:
Find $\frac{dy}{dx}$ for y defined implicitly by the equation:

$$4x^5 + \tan y = y^2 + 5x$$

Hint: Follow the problem-solving strategy, remembering to apply the chain rule to differentiate $\tan y$ and y^2.

- Step 1: Differentiate both sides of the equation with respect to x.
- Step 2: Apply the chain rule to differentiate $\tan y$ and y^2.
- Step 3: Collect terms involving $\frac{dy}{dx}$.
- Step 4: Solve for $\frac{dy}{dx}$.

Finding a Tangent Line to a Circle

Problem:
Find the equation of the line tangent to the curve $x^2 + y^2 = 25$ at the point $(3, -4)$.

Solution:

- **Step 1:** Recall that from implicit differentiation, we found the slope $\frac{dy}{dx} = \frac{-x}{y}$.
- **Step 2:** Substitute the point $(3, -4)$ into the slope equation:

$$\left.\frac{dy}{dx}\right|_{(3,-4)} = \frac{-3}{-4} = \frac{3}{4}$$

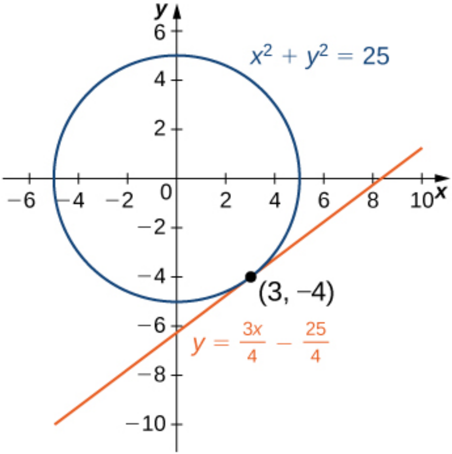
  The slope of the tangent line at $(3, -4)$ is $\frac{3}{4}$.
- **Step 3:** Use the point-slope form of the equation of a line, $y - y_1 = m(x - x_1)$, where m is the slope and (x_1, y_1) is the point $(3, -4)$:

$$y - (-4) = \frac{3}{4}(x - 3)$$

- **Step 4:** Simplify the equation: $y = \frac{3}{4}x - \frac{25}{4}$

Graph Interpretation

The circle $x^2 + y^2 = 25$, with radius 5 and center at the origin, has a tangent line at the point $(3, -4)$. The equation of the tangent line is $y = \frac{3}{4}x - \frac{25}{4}$.



Finding the Equation of the Tangent Line to a Curve

Problem:

Find the equation of the line tangent to the graph of $y^3 + x^3 - 3xy = 0$ at the point $\left(\frac{3}{2}, \frac{3}{2}\right)$. This curve is known as the folium of Descartes.

Solution:

- **Step 1:** Begin by finding $\frac{dy}{dx}$ using implicit differentiation:

$$\frac{d}{dx}\left(y^3 + x^3 - 3xy\right) = \frac{d}{dx}(0)$$

- **Step 2:** Differentiate each term:

$$3y^2\frac{dy}{dx} + 3x^2 - \left(3y + 3x\frac{dy}{dx}\right) = 0$$

- **Step 3:** Collect terms involving $\frac{dy}{dx}$:

$$3y^2\frac{dy}{dx} - 3x\frac{dy}{dx} = 3y - 3x^2$$

- **Step 4:** Factor out $\frac{dy}{dx}$:

Finding the Slope and Equation of the Tangent Line

Continuing the Solution:

- **Step 6:** Substitute the point $\left(\frac{3}{2}, \frac{3}{2}\right)$ into the derivative:

$$\left.\frac{dy}{dx}\right|_{\left(\frac{3}{2},\frac{3}{2}\right)} = \frac{3 \cdot \frac{3}{2} - 3 \cdot \left(\frac{3}{2}\right)^2}{3 \cdot \left(\frac{3}{2}\right)^2 - 3 \cdot \frac{3}{2}}$$

  Simplifying this expression:

$$\frac{dy}{dx} = \frac{\frac{9}{2} - \frac{27}{4}}{\frac{27}{4} - \frac{9}{2}} = -1$$

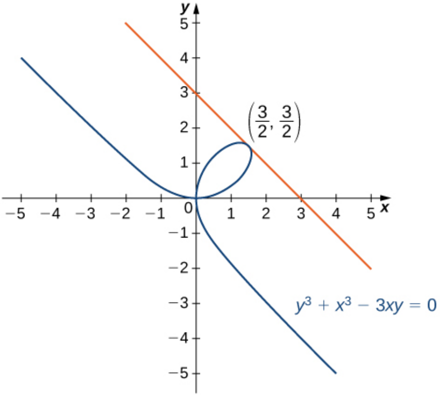
  The slope of the tangent line is -1.

- **Step 7:** Use the point-slope form of the equation of a line, $y - y_1 = m(x - x_1)$, with the point $\left(\frac{3}{2}, \frac{3}{2}\right)$ and slope -1:

$$y - \frac{3}{2} = -1\left(x - \frac{3}{2}\right)$$

- **Step 8:** Simplify the equation: $y = -x + 3$

Graph Description:
The graph below shows the folium of Descartes given by the equation $x^3 + y^3 - 3xy = 0$. The red line represents the tangent line at the point $\left(\frac{3}{2}, \frac{3}{2}\right)$, with the equation $y = -x + 3$.



Applying Implicit Differentiation: Tangent Line to a Rocket's Path

Problem:
In a simple video game, a rocket travels in an elliptical orbit whose path is described by the equation $4x^2 + 25y^2 = 100$. The rocket fires a missile at the point $\left(3, \frac{8}{3}\right)$. Where will the missile intersect the x-axis?

The task is to find the tangent line at this point and determine where the line intersects the x-axis.

Solution: Implicit Differentiation

Step 1: Begin by finding $\frac{dy}{dx}$ using implicit differentiation:

$$\frac{d}{dx}\left(4x^2 + 25y^2\right) = \frac{d}{dx}(100)$$

Differentiating both sides gives:

$$8x + 50y\frac{dy}{dx} = 0$$

Step 2: Solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{-4x}{25y}$$

Solution: Tangent Line Equation

Step 3: Substitute the point $\left(3, \frac{8}{3}\right)$ into the derivative:

$$\left.\frac{dy}{dx}\right|_{\left(3, \frac{8}{3}\right)} = \frac{-4 \cdot 3}{25 \cdot \frac{8}{3}} = \frac{-9}{50}$$

The slope of the tangent line is $\frac{-9}{50}$.

Step 4: Use the point-slope form to find the equation of the tangent line:

$$y - \frac{8}{3} = \frac{-9}{50}(x - 3)$$

Simplifying:

$$y = \frac{-9}{50}x + \frac{481}{150}$$

Solution: Intersection with the x-Axis

Step 5: To find the intersection with the x-axis, set $y = 0$:

$$0 = \frac{-9}{50}x + \frac{481}{150}$$

Solving for x:

$$x = \frac{481}{27}$$

Therefore, the missile intersects the x-axis at the point $\left(\frac{481}{27}, 0\right)$.

Tangent Line to the Hyperbola

Problem:
Find the equation of the line tangent to the hyperbola $x^2 - y^2 = 16$ at the point $(5, 3)$.

Hint: Using implicit differentiation, you should find that $\frac{dy}{dx} = \frac{x}{y}$.

Solution: Implicit Differentiation

Step 1: Differentiate both sides of the equation $x^2 - y^2 = 16$ implicitly with respect to x:

$$\frac{d}{dx}(x^2 - y^2) = \frac{d}{dx}(16)$$

This gives:

$$2x - 2y\frac{dy}{dx} = 0$$

Step 2: Solve for $\frac{dy}{dx}$:

$$2x = 2y\frac{dy}{dx} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{x}{y}$$

Solution: Finding the Tangent Line

Step 3: Substitute the point $(5, 3)$ into the equation for the slope:

$$\left.\frac{dy}{dx}\right|_{(5,3)} = \frac{5}{3}$$

The slope of the tangent line at $(5, 3)$ is $\frac{5}{3}$.

Step 4: Use the point-slope form of a line, $y - y_1 = m(x - x_1)$, to find the equation of the tangent line:

$$y - 3 = \frac{5}{3}(x - 5)$$

Step 5: Simplify the equation:

$$y = \frac{5}{3}x - \frac{16}{3}$$

Key Concepts

- **Implicit Differentiation:** We use implicit differentiation to find derivatives of implicitly defined functions, which are functions defined by equations involving both x and y (e.g., $x^2 - y^2 = 16$).
- **Tangent Line:** By using implicit differentiation, we can find the slope of the tangent line at any given point on the graph of a curve, allowing us to determine the equation of the tangent line.

Derivatives of Exponential and Logarithmic Functions

Clotilde Djuikem

Learning Objectives

- Find the derivative of all exponential functions.
- Find the derivative of logarithmic functions.
- Use logarithmic differentiation to determine the derivative of a function.

So far, we have learned how to differentiate a variety of functions, including trigonometric, inverse, and implicit functions. In this section, we explore derivatives of exponential and logarithmic functions.

Exponential functions

Exponential functions play an important role in modeling population growth and the decay of radioactive materials. Logarithmic functions can help rescale large quantities and are particularly helpful for rewriting complicated expressions.

Applying the Natural Exponential Function

A colony of mosquitoes has an initial population of 1000. After t days, the population is given by $A(t) = 1000e^{0.3t}$. Show that the ratio of the rate of change of the population, $A'(t)$, to the population size, $A(t)$, is constant.

Solution:

- Find $A'(t)$ using the chain rule:

$$A'(t) = 300e^{0.3t}$$

- The ratio of the rate of change to the population size is:

$$\frac{A'(t)}{A(t)} = \frac{300e^{0.3t}}{1000e^{0.3t}} = 0.3$$

Thus, the ratio is constant at 0.3.

Finding the Rate of Change of the Mosquito Population

Problem:
If $A(t) = 1000e^{0.3t}$ describes the mosquito population after t days, what is the rate of change of $A(t)$ after 4 days?

Hint:
To solve this, find $A'(4)$, the derivative of the population function evaluated at $t = 4$.

Solution

Solution:

- The population function is $A(t) = 1000e^{0.3t}$.
- To find the rate of change, we first differentiate $A(t)$:

$$A'(t) = 1000 \cdot 0.3e^{0.3t} = 300e^{0.3t}$$

- Now, substitute $t = 4$ into $A'(t)$:

$$A'(4) = 300e^{0.3 \cdot 4} = 300e^{1.2}$$

- Approximating $e^{1.2}$, we get:

$$A'(4) \approx 300 \cdot 3.3201 = 996$$

Therefore, the rate of change of the population after 4 days is approximately 996 mosquitoes per day.

Derivative of the Logarithmic Function

Now that we have the derivative of the natural exponential function, we can use implicit differentiation to find the derivative of its inverse, the natural logarithmic function.

Formula

If $x > 0$ and $y = \ln x$, then:

$$\frac{dy}{dx} = \frac{1}{x}$$

More generally, if $h(x) = \ln(g(x))$, then:

$$h'(x) = \frac{1}{g(x)} g'(x)$$

Proof of the Derivative of the Natural Logarithmic Function

Proof:

- If $x > 0$ and $y = \ln x$, then by exponentiating both sides we get:

$$e^y = x$$

- Differentiating both sides with respect to x, we apply the chain rule:

$$e^y \frac{dy}{dx} = 1$$

- Solving for $\frac{dy}{dx}$ gives:

$$\frac{dy}{dx} = \frac{1}{e^y}$$

- Since $e^y = x$, we substitute this into the equation:

$$\frac{dy}{dx} = \frac{1}{x}$$

Alternative Proof Using the Inverse Function Theorem

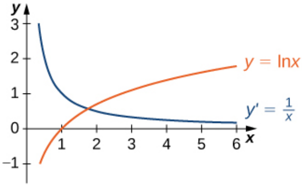
We may also derive this result by applying the inverse function theorem:

- Since $y = g(x) = \ln x$ is the inverse of $f(x) = e^x$, by the inverse function theorem, we have:

$$\frac{dy}{dx} = \frac{1}{f'(g(x))} = \frac{1}{e^{\ln x}} = \frac{1}{x}$$

- Using this result and applying the chain rule to $h(x) = \ln(g(x))$ yields:

$$h'(x) = \frac{1}{g(x)} g'(x)$$



Taking the Derivative of a Natural Logarithm

Find the derivative of $f(x) = \ln(x^3 + 3x - 4)$.

Solution: Use the chain rule:

$$f'(x) = \frac{1}{x^3 + 3x - 4} \cdot (3x^2 + 3)$$

Simplifying:

$$f'(x) = \frac{3x^2 + 3}{x^3 + 3x - 4}$$

Using Properties of Logarithms in a Derivative

Problem:
Find the derivative of $f(x) = \ln\left(\frac{x^2 \sin x}{2x+1}\right)$.

Solution:
At first glance, taking this derivative appears complicated. However, by using properties of logarithms before differentiating, we can simplify the problem:

Step 1: Apply the properties of logarithms:

$$f(x) = \ln\left(\frac{x^2 \sin x}{2x+1}\right) = 2\ln(x) + \ln(\sin x) - \ln(2x+1)$$

Solution Continued

Step 2: Differentiate using the sum rule:

$$f'(x) = \frac{2}{x} + \frac{\cos x}{\sin x} - \frac{2}{2x+1}$$

Step 3: Simplify the expression:

$$f'(x) = \frac{2}{x} + \cot x - \frac{2}{2x+1}$$

Conclusion: The derivative of $f(x)$ is:

$$f'(x) = \frac{2}{x} + \cot x - \frac{2}{2x+1}$$

Problem:
Differentiate $f(x) = \ln\left((3x+2)^5\right)$.

Hint:
Use a property of logarithms to simplify before taking the derivative.

Step 1: Simplify using the logarithm property $\ln(a^b) = b\ln(a)$:

$$f(x) = 5\ln(3x+2)$$

Solution: Differentiation

Step 2: Differentiate $f(x) = 5\ln(3x+2)$ using the chain rule:

$$f'(x) = 5 \cdot \frac{1}{3x+2} \cdot 3$$

Step 3: Simplify the expression:

$$f'(x) = \frac{15}{3x+2}$$

Conclusion: The derivative of $f(x)$ is:

$$f'(x) = \frac{15}{3x+2}$$

Derivatives of General Exponential and Logarithmic Functions

Let $b > 0$, $b \neq 1$, and let $g(x)$ be a differentiable function.

Logarithmic Function:

- If $y = \log_b x$, then:
$$\frac{dy}{dx} = \frac{1}{x \ln b}$$
- More generally, if $h(x) = \log_b(g(x))$, then:
$$h'(x) = \frac{g'(x)}{g(x) \ln b}$$

Exponential Function:

- If $y = b^x$, then:

$$\frac{dy}{dx} = b^x \ln b$$

- More generally, if $h(x) = b^{g(x)}$, then:

$$h'(x) = b^{g(x)} g'(x) \ln b$$

Applying Derivative Formulas: Quotient Rule

Problem:
Find the derivative of $h(x) = \frac{3^x}{3x+2}$.

Solution:
Use the quotient rule:

$$h'(x) = \frac{(3^x \ln 3)(3x+2) - 3^x \cdot 3}{(3x+2)^2}$$

Finding the Slope of a Tangent Line

Problem:
Find the slope of the tangent line to the graph of $y = \log_2(3x + 1)$ at $x = 1$.

Solution:
Step 1: Differentiate $y = \log_2(3x + 1)$. Using the derivative formula for logarithms with arbitrary base:

$$\frac{dy}{dx} = \frac{1}{(3x + 1)\ln 2} \cdot 3$$

Step 2: Simplify the expression for the derivative:

$$\frac{dy}{dx} = \frac{3}{(3x + 1)\ln 2}$$

Solution: Slope at $x = 1$

Step 3: Evaluate the derivative at $x = 1$:

$$\left.\frac{dy}{dx}\right|_{x=1} = \frac{3}{(3(1)+1)\ln 2} = \frac{3}{4\ln 2}$$

Conclusion: The slope of the tangent line to the graph at $x = 1$ is:

$$\frac{3}{4\ln 2}$$

Finding the Slope of the Tangent Line

Problem:
Find the slope of the tangent line to $y = 3^x$ at $x = 2$.

Hint:
Evaluate the derivative at $x = 2$.

Step 1: Differentiate $y = 3^x$. The derivative of 3^x is:

$$\frac{dy}{dx} = 3^x \ln 3$$

Solution: Slope at $x = 2$

Step 2: Evaluate the derivative at $x = 2$:

$$\left.\frac{dy}{dx}\right|_{x=2} = 3^2 \ln 3 = 9 \ln 3$$

Conclusion: The slope of the tangent line at $x = 2$ is:

$$9 \ln 3$$

Problem-Solving Strategy: Using Logarithmic Differentiation

Steps to Differentiate $y = h(x)$ using Logarithmic Differentiation:

1. Take the natural logarithm of both sides:

$$\ln y = \ln\left(h(x)\right)$$

2. Use properties of logarithms to expand $\ln\left(h(x)\right)$ as much as possible.
3. Differentiate both sides of the equation. On the left-hand side, we get:

$$\frac{1}{y}\frac{dy}{dx}$$

4. Multiply both sides of the equation by y to solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = y \cdot (\text{derivative of the right-hand side})$$

5. Finally, replace y with the original expression $h(x)$.

Using Logarithmic Differentiation

Problem:
Find the derivative of $y = (2x^4 + 1)^{\tan x}$.

Solution:
Use logarithmic differentiation to find the derivative.
Step 1: Take the natural logarithm of both sides:

$$\ln y = \ln\left((2x^4 + 1)^{\tan x}\right)$$

Step 2: Expand using properties of logarithms:

$$\ln y = \tan x \cdot \ln(2x^4 + 1)$$

Solution Continued

Step 3: Differentiate both sides. Using the product rule on the right-hand side:

$$\frac{1}{y}\frac{dy}{dx} = \sec^2 x \cdot \ln(2x^4 + 1) + \frac{8x^3}{2x^4 + 1} \cdot \tan x$$

Step 4: Multiply both sides by y to solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = y \cdot \left(\sec^2 x \cdot \ln(2x^4 + 1) + \frac{8x^3}{2x^4 + 1} \cdot \tan x\right)$$

Step 5: Substitute $y = (2x^4 + 1)^{\tan x}$:

$$\frac{dy}{dx} = (2x^4 + 1)^{\tan x} \left(\sec^2 x \cdot \ln(2x^4 + 1) + \frac{8x^3}{2x^4 + 1} \cdot \tan x\right)$$

Using Logarithmic Differentiation

Problem:
Find the derivative of $y = x\sqrt{2x+1}e^x \sin^3 x$.

Solution:
This problem makes use of the properties of logarithms and differentiation rules.
Step 1: Take the natural logarithm of both sides:

$$\ln y = \ln\left(x\sqrt{2x+1}e^x \sin^3 x\right)$$

Step 2: Use properties of logarithms to expand:

$$\ln y = \ln x + \frac{1}{2}\ln(2x+1) + \ln e^x + 3\ln(\sin x)$$

Solution Continued

Step 3: Differentiate both sides:

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x} + \frac{1}{2x+1} \cdot 2 - 1 + 3 \cdot \frac{\cos x}{\sin x}$$

Simplifying:

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x} + \frac{1}{2x+1} - 1 - 3\cot x$$

Step 4: Multiply both sides by y to solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = y\left(\frac{1}{x} + \frac{1}{2x+1} - 1 - 3\cot x\right)$$

Solution Final Step

Step 5: Substitute $y = x\sqrt{2x+1}e^x \sin^3 x$:

$$\frac{dy}{dx} = x\sqrt{2x+1}e^x \sin^3 x \left(\frac{1}{x} + \frac{1}{2x+1} - 1 - 3\cot x \right)$$

Final Answer:
The derivative of $y = x\sqrt{2x+1}e^x \sin^3 x$ is:

$$\frac{dy}{dx} = x\sqrt{2x+1}e^x \sin^3 x \left(\frac{1}{x} + \frac{1}{2x+1} - 1 - 3\cot x \right)$$

Problem:
Find the derivative of $y = x^r$, where r is an arbitrary real number.

Solution:
The process follows the steps of logarithmic differentiation.
Step 1: Take the natural logarithm of both sides:

$$\ln y = \ln(x^r)$$

Step 2: Use properties of logarithms to expand:

$$\ln y = r \ln x$$

Solution Continued

Step 3: Differentiate both sides:

$$\frac{1}{y}\frac{dy}{dx} = r \cdot \frac{1}{x}$$

Step 4: Multiply both sides by y to solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = y \cdot \frac{r}{x}$$

Step 5: Substitute $y = x^r$ into the equation:

$$\frac{dy}{dx} = x^r \cdot \frac{r}{x}$$

Simplifying:

$$\frac{dy}{dx} = rx^{r-1}$$

Using Logarithmic Differentiation

Problem:
Find the derivative of $y = x^x$.

Hint:
Follow the problem-solving strategy using logarithmic differentiation.
Step 1: Take the natural logarithm of both sides:

$$\ln y = \ln\left(x^x\right)$$

Step 2: Use properties of logarithms to expand:

$$\ln y = x \ln x$$

Solution for $y = x^x$

Step 3: Differentiate both sides:

$$\frac{1}{y}\frac{dy}{dx} = \ln x + 1$$

Step 4: Multiply both sides by y to solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = y(\ln x + 1)$$

Step 5: Substitute $y = x^x$:

$$\frac{dy}{dx} = x^x(\ln x + 1)$$

Finding the Derivative of $y = (\tan x)^\pi$

Problem:
Find the derivative of $y = (\tan x)^\pi$.

Hint:
Use the result from the general power rule for logarithmic differentiation.

Step 1: Differentiate $y = (\tan x)^\pi$. Since π is constant, we apply the power rule:

$$y' = \pi(\tan x)^{\pi-1} \cdot \sec^2 x$$

Conclusion: The derivative of $y = (\tan x)^\pi$ is:

$$y' = \pi(\tan x)^{\pi-1} \sec^2 x$$

Key Concepts

1. Differentiability of Exponential Functions:

- The exponential function $y = b^x$, where $b > 0$, is continuous everywhere and differentiable at 0.
- This implies that $y = b^x$ is differentiable everywhere, and we can use the formula for its derivative:

$$\frac{dy}{dx} = b^x \ln b$$

2. Derivative of the Natural Logarithmic Function:

- We can find the derivative of $y = \ln x$ using:

$$\frac{dy}{dx} = \frac{1}{x}$$

- The relationship $\log_b x = \frac{\ln x}{\ln b}$ allows us to extend differentiation formulas to logarithms with arbitrary bases.

Key Concepts Continued

3. Logarithmic Differentiation:

- Logarithmic differentiation is a useful technique for differentiating functions of the form $y = g(x)^{f(x)}$.
- It is especially helpful for differentiating complex functions by taking the natural logarithm of both sides and exploiting properties of logarithms before differentiating.
- Example:

$$\ln y = f(x)\ln(g(x)) \quad \Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = f'(x)\ln g(x) + f(x)\frac{g'(x)}{g(x)}$$

- After differentiation, multiply both sides by y to solve for $\frac{dy}{dx}$.

Key Equations

1. **Derivative of the Natural Exponential Function:**

$$\frac{d}{dx}\left(e^{g(x)}\right) = e^{g(x)}g'(x)$$

2. **Derivative of the Natural Logarithmic Function:**

$$\frac{d}{dx}(\ln(g(x))) = \frac{1}{g(x)}g'(x)$$

Key Equations Continued

3. Derivative of the General Exponential Function:

$$\frac{d}{dx}\left(b^{g(x)}\right) = b^{g(x)}g'(x)\ln b$$

4. Derivative of the General Logarithmic Function:

$$\frac{d}{dx}\left(\log_b(g(x))\right) = \frac{g'(x)}{g(x)\ln b}$$