Math 1500 - A06, Fall 2024
Introduction to Functions

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Think About This

Computer Science

How can we speed up a program with a simple idea?

Agribusiness

How can we predict next season's crop yield?

Science & Agriculture

How can we model environmental changes?

Arts

How can we find hidden trends in cultural data?

Economics

How can we analyze market behavior and predict economic trends?

The Answer Is Simple

All of these questions can be answered using a key mathematical concept:

Functions

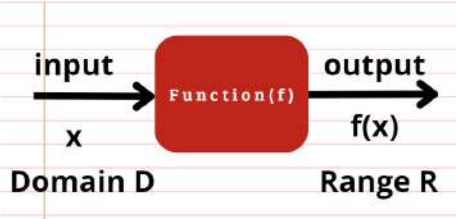
What is a Function?

Definition

A function consists of a set of inputs, a set of outputs, and a rule for assigning each input to exactly one output. The **set of inputs is called the domain** of the function. The **set of outputs is called the range** of the function.

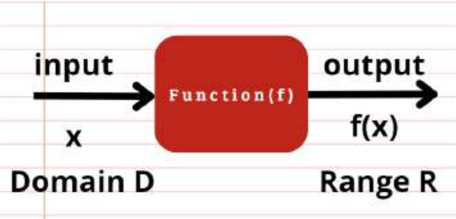
Example: Imagine a vending machine where you press a button (input) and get a specific snack (output). The machine always gives you the same snack for the same button.

Domain and range

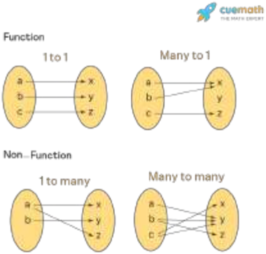


Function vs Non-Function: What's the Difference?

Domain and range



Function vs non-function?



Function or Not? Let's Analyze!

Example 1:

$$f(x) = 2x + 3$$

Is this a function? Yes
Explanation: For each input x, there is exactly one output $f(x)$. This satisfies the definition of a function.

Example 2:

$$f(x) = \sqrt{x}$$

Is this a function? Yes
Explanation: For non-negative values of x, this function assigns exactly one output $f(x)$, making it a valid function.

Example 3:

$$f(x) = \pm\sqrt{x}$$

Is this a function? No
Explanation: The same input x can produce two outputs, $+\sqrt{x}$ and $-\sqrt{x}$, which violates the definition of a function.

Example 4:

$$f(x) = \frac{1}{x}$$

Is this a function? Yes
Explanation: For all $x \neq 0$, each input produces exactly one output, so it is a valid function.

The Domain of a Function

Definition

The domain of a function f is the set of all possible input values x for which the function $f(x)$ is defined.

Examples:

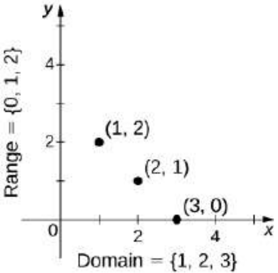
- **Example 1:** $f(x) = 2x + 3$
  Domain: $\mathbb{R} = (-\infty, \infty) = \{x | x \text{ is any real number}\}$ (all real numbers)
  Explanation: The function is defined for all real values of x, since there are no restrictions like square roots or divisions by zero.
- **Example 2:** $f(x) = \frac{1}{x}$
  Domain: $\mathbb{R} \setminus \{0\}$
  Explanation: The function is not defined when $x = 0$ because division by zero is undefined.
- **Example 3:** $f(x) = \sqrt{x}$
  Domain: $[0, \infty) = \{x | 0 \leq x\}$
  Explanation: The function is only defined for $x \geq 0$, because the square root of a negative number is not a real number.

The Graph of a function

Definition

The graph of a function f is the set of all points $(x, f(x))$ in the plane where x belongs to the domain of the function.

Example: The graph of $f(x) = 3 - x$ is a straight line, since for each x, the output is $f(x)$.



Learning Objectives

- Calculate the slope of a linear function and interpret its meaning.
- Recognize the degree of a polynomial.
- Find the roots of a quadratic polynomial.
- Describe the graphs of basic odd and even polynomial functions.
- Identify a rational function.
- Describe the graphs of power and root functions.
- Explain the difference between algebraic and transcendental functions.
- Graph a piecewise-defined function.
- Sketch the graph of a function that has been shifted, stretched, or reflected.

Linear Functions and Slope

Definition of a Linear Function:

$$f(x) = mx + b$$

Where:

- m is the slope of the line.
- b is the y-intercept (where the line crosses the y-axis).

Slope:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

The slope represents the rate of change, or how much y changes for each unit change in x.

Interpretation of Slope:

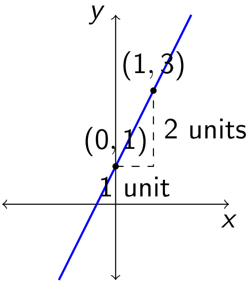
- If $m > 0$, the line is increasing (rising as x increases).
- If $m < 0$, the line is decreasing (falling as x increases).
- If $m = 0$, the line is horizontal.

Linear functions and slope: Example

Example: Let's consider the linear function:

$$f(x) = 2x + 1$$

Slope: $m = 2$ (the line rises by 2 for each 1 unit increase in x)
y-intercept: $b = 1$



Interpretation:

- The slope $m = 2$ means that for every 1 unit increase in x, y increases by 2 units.
- The y-intercept is the point where the line crosses the y-axis at $(0, 1)$.

Standard form

Important

The standard form of a line is given by the equation:

$$ax + by = c$$

where a and b are both non-zero. This form is more general because it allows for a vertical line, $x = k = \frac{c}{a}$, when $b = 0$.

Point-Slope Form

Consider a line passing through the point (x_1, y_1) with slope m. The equation:

$$y - y_1 = m(x - x_1)$$

is the point-slope equation for that line.

Example: Finding the Equation of a Line

Example: Given the points $(2, 3)$ and $(4, 7)$, find the equation of the line passing through these points.
Step 1: Calculate the slope m

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{7 - 3}{4 - 2} = \frac{4}{2} = 2$$

Step 2: Use the point-slope form $y - y_1 = m(x - x_1)$ with $m = 2$ and $(x_1, y_1) = (2, 3)$

$$y - 3 = 2(x - 2)$$

Step 3: Simplify the equation

$$y - 3 = 2x - 4$$

$$y = 2x - 1$$

Thus, the equation of the line passing through the points $(2, 3)$ and $(4, 7)$ is:

$$y = 2x - 1$$

Polynomials

Definition

A polynomial function can be written as:

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

Where:

- n is the degree (highest power of x).
- a_n is the leading coefficient.

Special Cases:

- Degree 1: Linear function $f(x) = mx + b$
- Degree 2: Quadratic function $f(x) = ax^2 + bx + c$
- Degree 3: Cubic function $f(x) = ax^3 + bx^2 + cx + d$

Power Functions

A power function has the form:

$$f(x) = ax^n$$

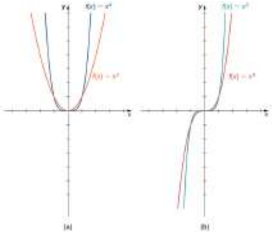
Where a and n are real numbers.

Even vs. Odd Functions:

- If n is even, the function is symmetric (even function $f(-x) = f(x)$).
- If n is odd, the function is antisymmetric (odd function $f(-x) = -f(x)$).

Example:

$$f(x) = x^2 \quad \text{(Even)}, \quad f(x) = x^3 \quad \text{(Odd)}$$



Behavior at Infinity

As $x \to \infty$, observe the behavior of polynomial functions:

- For $f(x) = ax^2 + bx + c$ (quadratic): If $a > 0$, $f(x) \to \infty$; if $a < 0$, $f(x) \to -\infty$.
- For $f(x) = ax^3 + bx^2 + cx + d$ (cubic): As $x \to \infty$, $f(x) \to \infty$ if $a > 0$, and $f(x) \to -\infty$ if $a < 0$.

Example:

$$f(x) = 3x^2 \quad \text{as} \quad x \to \infty, \quad f(x) \to \infty$$

Zeros of Polynomial Functions

The zeros of a polynomial function are where it intersects the x-axis. This is found by solving $f(x) = 0$.

Example 1: Linear Function For a linear function $f(x) = mx + b$, the x-intercept is:

$$x = \frac{-b}{m}$$

Example: If $f(x) = 2x - 6$, the x-intercept is:

$$x = \frac{-(-6)}{2} = 3$$

Example 2: Quadratic Function For a quadratic function $f(x) = ax^2 + bx + c$, solve:

$$ax^2 + bx + c = 0$$

Using the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Understanding the Discriminant

The term under the square root in the quadratic formula, $\Delta = b^2 - 4ac$, is called the **discriminant**. It determines the number and type of solutions for the quadratic equation:

$$x = \frac{-b \pm \sqrt{\Delta}}{2a}$$

Three Cases:

- If $\Delta > 0$, there are two distinct real solutions.
- If $\Delta = 0$, there is one real solution (repeated root).
- If $\Delta < 0$, there are no real solutions (complex solutions).

Example: Solve $2x^2 - 4x - 6 = 0$

Here, $a = 2$, $b = -4$, $c = -6$

$$\Delta = (-4)^2 - 4(2)(-6) = 16 + 48 = 64$$

Since $\Delta > 0$, there are two real solutions:

$$x = \frac{4 \pm \sqrt{64}}{4} = \frac{4 \pm 8}{4}$$ **Thus,** $x = 3$ **or** $x = -1$.

Step-by-Step Analysis of $f(x) = 2x^2 - 4x - 6$

1. Finding the Zeros: The zeros are $x = 3$ and $x = -1$. These are the points where the graph crosses the x-axis.
2. Behavior at Infinity:

$$\lim_{x \to \infty} f(x) = \infty \quad \text{and} \quad \lim_{x \to -\infty} f(x) = \infty$$

Since the leading term $2x^2$ dominates, the graph rises to infinity as $x \to \pm\infty$.
3. Vertex: The x-coordinate of the vertex is given by:

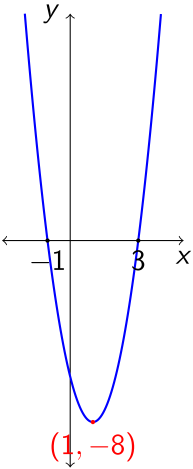
$$x = \frac{-b}{2a} = \frac{-(-4)}{2(2)} = 1$$

Substituting $x = 1$ into the function to find y:

$$f(1) = 2(1)^2 - 4(1) - 6 = -8$$

The vertex is at $(1, -8)$.

Graph of $f(x) = 2x^2 - 4x - 6$



Key Features:

- Zeros at $x = 3$ and $x = -1$
- Vertex at $(1, -8)$
- The graph rises to infinity as $x \to \infty$ and $x \to -\infty$

Mathematical models: Definition

A mathematical model uses equations and functions to simulate real-world situations. It helps predict outcomes based on data.

Uses of Mathematical Models:

- Population growth prediction
- Weather forecasting
- Business revenue prediction

Example: Revenue Model

A company models its revenue $R(p)$, where p is the price per item and $n(p)$ is the number of units sold, using a function:

$$R(p) = p \cdot n(p)$$

How to Maximize Revenue?

Creating the Revenue Model

The company uses data to model the number of units sold as a linear function of price:

$$n(p) = -1.04p + 26$$

Table of Prices and Units Sold (in Thousands):

Price per Unit ($)	Units Sold (Thousands)
6	19.4
8	18.5
10	16.2
12	13.8
14	12.2

Revenue Function: Substitute $n(p)$ into the revenue equation:

$$R(p) = p \cdot (-1.04p + 26) = -1.04p^2 + 26p$$

This quadratic equation models the company's revenue as a function of the price per item.

Maximizing Revenue

Step 1: Find the Zeros of $R(p)$:

$$R(p) = -1.04p^2 + 26p = 0$$

Factoring gives:

$$p(-1.04p + 26) = 0 \quad \Rightarrow \quad p = 0 \quad \text{or} \quad p = 25$$

These zeros mean that at prices of \$0 and \$25, the revenue is zero.
Step 2: Find the Price that Maximizes Revenue: The price that maximizes revenue is found using the vertex formula for a parabola:

$$p_{\max} = \frac{-b}{2a} = \frac{-26}{2(-1.04)} = 12.5$$

Step 3: Calculate the Maximum Revenue:

$$R(12.5) = -1.04(12.5)^2 + 26(12.5) = 162.5 \quad \text{(in thousands)}$$

Conclusion: The maximum revenue is \$162,500 when the price is \$12.50 per item.

Algebraic Functions

An algebraic function involves addition, subtraction, multiplication, division, and powers of variables. Two common types of algebraic functions are:

- **Rational Functions:** Quotients of polynomials.

  $$f(x) = \frac{p(x)}{q(x)}$$

  Example:

  $$f(x) = \frac{3x - 1}{5x + 2}$$

- **Root Functions:** Functions that involve roots of variables.

  $$f(x) = x^{1/n}$$

  Example:

  $$f(x) = \sqrt{x} \quad \text{(Square Root Function)}$$

These functions form a larger class of functions that can describe various real-world situations.

Finding Domain and Range of Algebraic Functions

Example 1: Rational Function

$$f(x) = \frac{3x-1}{5x+2}$$

Domain: The denominator cannot be zero.

$$5x + 2 \neq 0 \quad \Rightarrow \quad x \neq -\frac{2}{5}$$

Range: Solve for $y = \frac{3x-1}{5x+2}$ and find values of y for which the equation has a solution.

Example 2: Root Function

$$f(x) = \sqrt{4 - x^2}$$

Domain: The expression under the square root must be non-negative.

$$4 - x^2 \geq 0 \quad \Rightarrow \quad -2 \leq x \leq 2$$

Range: The range of $f(x)$ is $0 \leq y \leq 2$.

Characteristics of Root Functions

Even Root Functions: For even integers $n \geq 2$:

$$f(x) = x^{1/n} \quad \text{Domain:}[0, \infty)$$

Example:

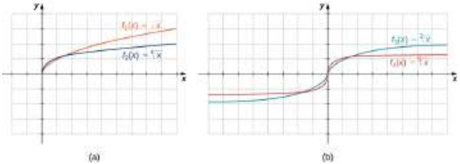
$$f(x) = \sqrt{x}$$

Odd Root Functions: For odd integers $n \geq 1$:

$$f(x) = x^{1/n} \quad \text{Domain:}\mathbb{R}$$

Examples:

$$f(x) = \sqrt[3]{x}$$



Finding Domains for Algebraic Functions

For each of the following functions, determine the domain:

$$f(x) = \frac{3x^2 - 1}{x^2 - 1} \quad f(x) = \frac{2x + 5}{3x^2 + 4} \quad f(x) = \sqrt{4 - 3x} \quad f(x) = \sqrt[3]{2x - 1}$$

Finding Domains for Algebraic Functions

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Solutions:

- $f(x) = \frac{3x^2-1}{x^2-1}$

$$x^2 - 1 \neq 0 \quad \Rightarrow \quad x \neq \pm 1$$

  Domain: $\{x \mid x \neq \pm 1\}$

- $f(x) = \frac{2x+5}{3x^2+4}$

$$3x^2 + 4 \geq 4 \quad \text{for all} \quad x \in \mathbb{R}$$

  Domain: $(-\infty, \infty)$

- $f(x) = \sqrt{4 - 3x}$

$$4 - 3x \geq 0 \quad \Rightarrow \quad x \leq \frac{4}{3}$$

  Domain: $\{x \mid x \leq \frac{4}{3}\}$

- $f(x) = \sqrt[3]{2x - 1}$

Cube roots are defined for all real numbers.

Transcendental Functions

Transcendental functions go beyond algebraic operations. The most common transcendental functions are:

- **Trigonometric Functions:** $\sin x, \cos x, \tan x, \cot x, \sec x, \csc x$
- **Exponential Functions:** $f(x) = b^x$
- **Logarithmic Functions:** $f(x) = \log_b(x)$

Classifying Functions:

- $f(x) = \sqrt{x^3 + \frac{1}{4}x + 2}$ (Algebraic Function)
- $f(x) = 2^x$ (Transcendental Function)
- $f(x) = \sin(2x)$ (Transcendental Function)

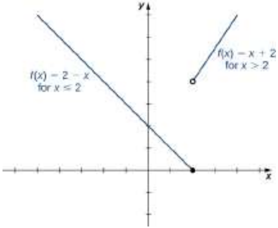
Piecewise-Defined Functions

A piecewise-defined function is a function defined by different formulas over different parts of its domain. For example:

$$f(x) = \begin{cases} x + 2, & \text{if } x < 2 \\ 2 - X, & \text{if } x \geq 2 \end{cases}$$

To graph:

- Graph $f(x) = x + 2$ for $x < 2$.
- Graph $f(x) = 2 - X$ for $x \geq 2$.
- Use an open circle for $(2, 4)$ and a closed circle for $(2, 0)$.



Piecewise Function Example – Parking Fees

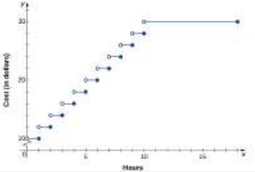
In a parking garage, the cost $C(x)$ is defined as:

$$C(x) = \begin{cases} 10, & \text{if } 0 < x \leq 1 \\ 12, & \text{if } 1 < x \leq 2 \\ \vdots & \\ 30, & \text{if } 10 < x \leq 18 \end{cases}$$

The graph consists of horizontal line segments where each segment corresponds to an interval of hours parked.

Graph:

- Horizontal segments increase by 2 for each additional hour.
- The maximum cost is $30 after 10 hours.



Transformations of Functions

Transformations change the appearance of a graph in several ways:

- **Vertical Shift:** Adding/subtracting a constant moves the graph up or down.

$$f(x) + c \quad \text{(shift up)} \quad f(x) - c \quad \text{(shift down)}$$

- **Horizontal Shift:** Adding/subtracting to x moves the graph left or right.

$$f(x - c) \quad \text{(shift right)} \quad f(x + c) \quad \text{(shift left)}$$

- **Vertical Scaling:** Multiplying the function by c stretches or compresses it.

$$cf(x) \quad \text{(stretch if } c > 1\text{, compress if } 0 < c < 1\text{)}$$

- **Reflections:** Multiplying by -1 flips the graph.

$$-f(x) \quad \text{(reflection over } x\text{-axis)} \quad f(-x) \quad \text{(reflection over } y\text{-axis)}$$

Examples: Shifts and Reflections

Example 1: Vertical Shift

$$f(x) = x^2, \quad f(x) + 3 \quad \text{(shift up 3 units)}$$

Graph $f(x) = x^2$ is shifted upward by 3 units.
Example 2: Horizontal Shift

$$f(x) = |x|, \quad f(x+2) \quad \text{(shift left 2 units)}$$

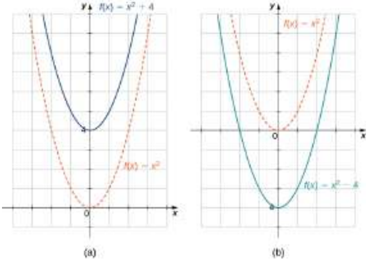
Graph $f(x) = |x|$ is shifted left by 2 units.
Example 3: Reflection

$$f(x) = x^3, \quad -f(x) = -x^3 \quad \text{(reflection over } x\text{-axis)}$$

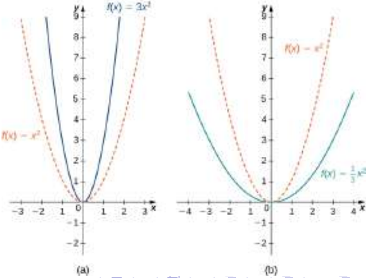
Graph $f(x) = x^3$ is reflected over the x-axis.

Vertical Shifts of $f(x) = x^2$ and $f(x) = 3x^2$

- Graph (a): $f(x) = x^2 + 4$ (Shifted up by 4 units)
- Graph (b): $f(x) = x^2 - 4$ (Shifted down by 4 units)



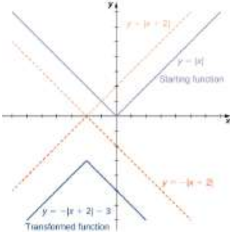
- Graph (a): $f(x) = 3x^2$ (Vertically stretched)
- Graph (b): $f(x) = \frac{1}{3}x^2$ (Vertically compressed)



Graphing Transformations of Piecewise Functions

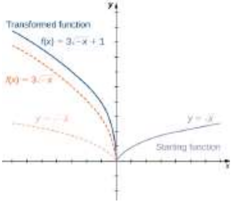
Function (a): $f(x) = -|x+2| - 3$

- Start with $y = |x|$
- Shift left 2 units: $y = |x+2|$
- Reflect over the x-axis: $y = -|x+2|$
- Shift down 3 units: $y = -|x+2| - 3$



Function (b): $f(x) = 3\sqrt{-x} + 1$

- Start with $y = \sqrt{x}$
- Reflect over the y-axis: $y = \sqrt{-x}$
- Stretch vertically by 3: $y = 3\sqrt{-x}$
- Shift up 1 unit: $y = 3\sqrt{-x} + 1$



Key Concepts

- A power function $f(x) = x^n$ is:
  - **Even** if n is even
  - **Odd** if n is odd
- The **root function** $f(x) = x^{1/n}$ has the domain:
  - $[0, \infty)$ if n is even
  - $(-\infty, \infty)$ if n is odd
- For a **rational function** $f(x) = \frac{p(x)}{q(x)}$, the domain is where $q(x) \neq 0$.
- **Algebraic functions** involve basic operations like addition, subtraction, multiplication, and division.
- **Transcendental functions** include trigonometric, exponential, and logarithmic functions.
- **Polynomial functions** $f(x) \to \pm\infty$ as $x \to \pm\infty$, depending on the leading term.
- **Transformations** include:
  - Shifts (left/right, up/down)
  - Stretching/compressing
  - Reflections (over the x- or y-axis)

Inverse Functions

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Inverse Functions

Why Are Inverse Functions Important?

In the real world, we often face problems where we know the result but need to figure out the starting point. Inverse functions help us solve such problems.

Examples:

- **Finance:** You know the final price after tax or discount, but what was the original price?
- **Engineering:** You have the distance traveled by an object, but what was its initial velocity?
- **Medicine:** A dosage of medication produces a certain effect, but what was the required dose?

Inverse functions allow us to reverse these processes and find the unknowns. In our next lesson, we explore how to work with inverse functions.

Learning Objectives

At the end of this lesson, you will be able to:

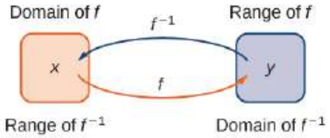
- Determine the conditions for when a function has an inverse.
- Use the horizontal line test to recognize one-to-one functions.
- Find the inverse of a given function.
- Graph the inverse of a function.
- Evaluate inverse trigonometric functions.

What is an Inverse Function?

Definition

An inverse function reverses the operation of a function. If $f(x)$ transforms an input x into an output y, then $f^{-1}(y)$ transforms the output y back into the original input x.

$$f^{-1}(f(x)) = x \quad \text{for all} \quad x \in D, \quad \text{and} \quad f(f^{-1}(y)) = y \quad \text{for all} \quad y \in R.$$



- If $f(x) = x^3 + 4$, then $f^{-1}(y) = \sqrt[3]{y-4}$.
- Inverse functions "undo" what the original function does.

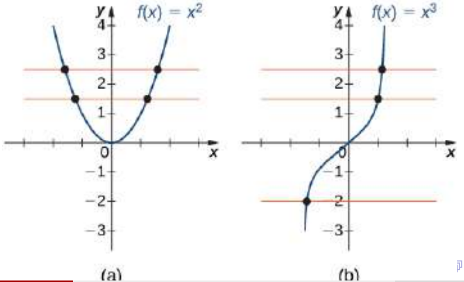
Conditions for Inverse Functions

One-to-One Functions

For a function to have an inverse, it must be one-to-one. This means that each output corresponds to exactly one input.

$$f(x_1) \neq f(x_2) \quad \text{when} \quad x_1 \neq x_2.$$

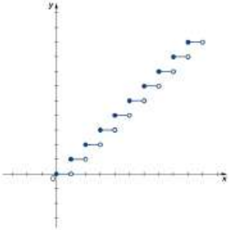
- Use the **horizontal line test**: If any horizontal line crosses the graph more than once, the function is not one-to-one, and it does not have an inverse.



Horizontal Line Test: One-to-One Functions

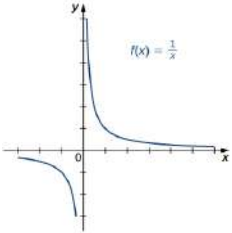
Step Function:

- The graph has multiple horizontal steps.
- Each step spans a constant value over the x-axis.
- **Conclusion:** This function is **not one-to-one**, as a horizontal line intersects the graph at multiple points.



Reciprocal Function $f(x) = \frac{1}{x}$:

- The graph continuously decreases in both quadrants.
- Horizontal lines intersect the graph at only one point.
- **Conclusion:** This function is **one-to-one**.



How to Find an Inverse Function

Steps to Find the Inverse:

- Start with $y = f(x)$.
- Solve the equation for x.
- Swap x and y and write $y = f^{-1}(x)$.

Example: For $f(x) = 3x - 4$,

$$y = 3x - 4 \quad \Rightarrow \quad x = \frac{y+4}{3} \quad \Rightarrow \quad f^{-1}(x) = \frac{x+4}{3}$$

How to Find an Inverse Function

Steps to Find the Inverse:

- Start with $y = f(x)$.
- Solve the equation for x.
- Swap x and y and write $y = f^{-1}(x)$.

Example: For $f(x) = \sqrt{x-2}$,

$$y = \sqrt{x-2} \quad \Rightarrow \quad y^2 = x - 2 \quad \Rightarrow \quad x = y^2 + 2$$

Therefore, the inverse function is:

$$f^{-1}(x) = x^2 + 2$$

How to Find an Inverse Function

Steps to Find the Inverse:

- Start with $y = f(x)$.
- Solve the equation for x.
- Swap x and y and write $y = f^{-1}(x)$.

Example: For $f(x) = \frac{2x+3}{x-1}$,

$$y = \frac{2x+3}{x-1} \quad \Rightarrow \quad y(x-1) = 2x+3 \quad \Rightarrow \quad yx - y = 2x+3$$

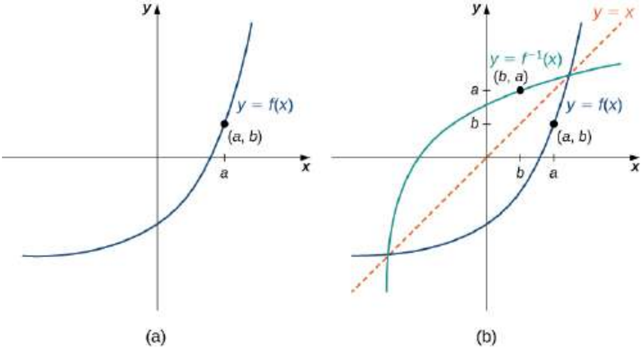
$$yx - 2x = y+3 \quad \Rightarrow \quad x(y-2) = y+3 \quad \Rightarrow \quad x = \frac{y+3}{y-2}$$

Therefore, the inverse function is:

$$f^{-1}(x) = \frac{x+3}{x-2}$$

Graphing an Inverse Function

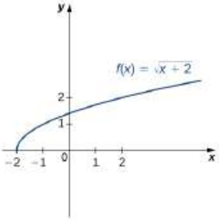
- The graph of an inverse function is the reflection of the original function over the line $y = x$.
- To graph the inverse, reflect each point (a, b) of $f(x)$ to (b, a) for $f^{-1}(x)$.



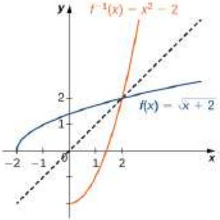
Solution:

- The graph of $f(x) = \sqrt{x+2}$ starts at the point $(-2, 0)$ and increases.
- The inverse function $f^{-1}(x) = x^2 - 2$ starts at the point $(0, -2)$.
- Use symmetry about the line $y = x$ to reflect the graph of f and obtain f^{-1}.
- The domain of f^{-1} is $[0, \infty)$ and its range is $[-2, \infty)$.

Graph of $f(x) = \sqrt{x+2}$



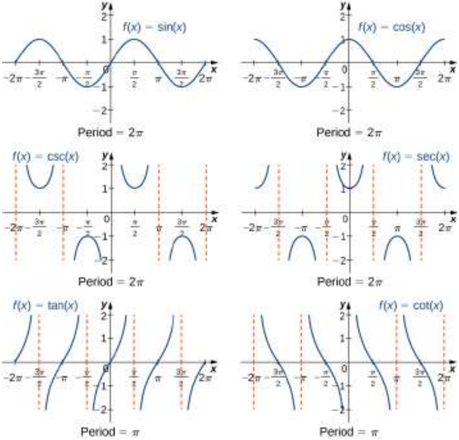
Graph of $f^{-1}(x) = x^2 - 2$



Key Point: The graph of $f(x)$ and $f^{-1}(x)$ are mirror images about the line $y = x$.

Inverse Trigonometric Functions

The six basic trigonometric functions are periodic, meaning they are not one-to-one.



However, by restricting the domain of a trigonometric function to an interval where it is one-to-one, we can define its inverse.

Definition:

- $\sin^{-1}(x) = y$ if and only if $\sin(y) = x$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$
- $\cos^{-1}(x) = y$ if and only if $\cos(y) = x$ and $0 \leq y \leq \pi$
- $\tan^{-1}(x) = y$ if and only if $\tan(y) = x$ and $-\frac{\pi}{2} < y < \frac{\pi}{2}$

Example:

Evaluate $\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$.

$$\sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$$

Key Concepts and Equations

Key Concepts

- For a function to have an inverse, it must be one-to-one.
- We can determine if a function is one-to-one using the horizontal line test.
- If a function is not one-to-one, we can restrict its domain to make it one-to-one and define its inverse on the restricted domain.
- For a function f and its inverse f^{-1},

$$f(f^{-1}(x)) = x \quad \text{for all} \quad x \quad \text{in the domain of} \quad f^{-1},$$
$$f^{-1}(f(x)) = x \quad \text{for all} \quad x \quad \text{in the domain of} \quad f.$$

- Graph of a function and its inverse are symmetric about $y = x$.
- Since trigonometric functions are periodic, we need to restrict their domains to define their inverses.

Key Equations

- Inverse functions:

$$f^{-1}(f(x)) = x \quad \text{for all} \quad x \in D, \quad \text{and} \quad f(f^{-1}(y)) = y \quad \text{for all} \quad y \in R.$$

Exponential and Logarithmic Functions

Clotilde Djuikem

Example: Money in the Bank

Let's imagine you start with an initial deposit of $C_0 = \$1000$ in the bank, and it grows at an annual interest rate of 2%.

After 1 year:

$$C_1 = C_0 + 0.02C_0 = 1000 + 0.02 \times 1000 = 1020$$

After 2 years:

$$C_2 = C_1 + 0.02C_1 = 1020 + 0.02 \times 1020 = 1040.40$$

After 3 years:

$$C_3 = C_2 + 0.02C_2 = 1040.40 + 0.02 \times 1040.40 = 1061.21$$

In general:

But if someone asks you what you will have after 50 years, would you calculate in the same way up to 50?

Think about the same problem with this approach

The initial deposit of $C_0 = C(0) = 1000\$$ and the rate is $r = 2\%$.
The amount in the bank after 1 year is:

$$C_1 = C(1) = C_0 + rC_0 = C_0(1+r)$$

The amount in the bank after 2 years is:

$$C(2) = C(1) + rC(1) = C(1) \times (1+r) = C_0(1+r)(1+r) = C_0(1+r)^2$$

In general: After t years, the amount in the bank is:

$$C(t) = C_0(1+r)^t$$

which is an exponential function

Then for the previous example

$$C(50) = 1000(1+0.02)^{50} = 2691.59\$$$

Exponential vs. Power Functions

Power Function: $y = x^b$

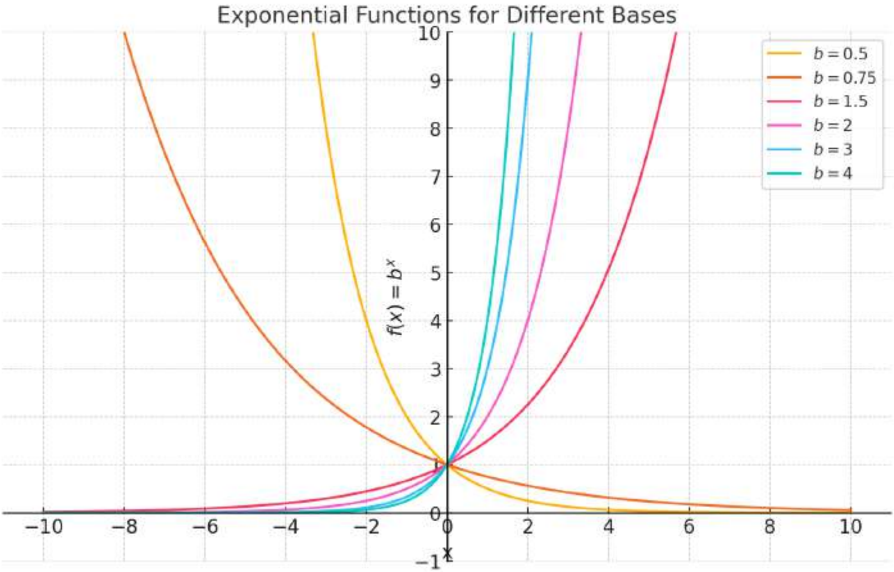
- Base: x (variable), Exponent: b (constant)
- As $x \to \infty$, $x^b \to \infty$ (for $b > 0$)
- As $x \to -\infty$, $x^b \to \infty$ (for $b > 0$)
- Symmetrical growth for positive and negative values of x

Exponential Function: $y = b^x$

- Base: b (constant), Exponent: x (variable)
- As $x \to \infty$, $b^x \to \infty$ (for $b > 1$)
- As $x \to -\infty$, $b^x \to 0$ (for $b > 1$)
- Exponential functions are always positive, $b^x > 0$ for all x
- Rapid growth for positive x, with a horizontal asymptote at $y = 0$

Although both functions approach infinity as $x \to \infty$, b^x (with a constant base) grows much faster than x^b (with a constant exponent). In contrast, x^b grows symmetrically for negative values, while b^x approaches zero as $x \to -\infty$.

Graph of Exponential Functions $f(x) = b^x$



Laws of Exponents

General Rules

For any constants $a \neq 0$, $b \neq 0$, and for all x and y:

- Multiplication: $b^x \cdot b^y = b^{x+y}$
- Division: $\frac{b^x}{b^y} = b^{x-y}$
- Power of a Power: $(b^x)^y = b^{xy}$
- Product of Powers: $(ab)^x = a^x b^x$
- Quotient of Powers: $\frac{a^x}{b^x} = \left(\frac{a}{b}\right)^x$

$$\textbf{Simplify } A = \frac{(2x^{2/3})^3}{(4x^{-1/3})^2} \textbf{ and } B = \frac{(x^3y^{-1})^2}{(xy^2)^{-2}}$$

Step 1: Distribute the exponents

$$(2x^{2/3})^3 = 2^3 \cdot (x^{2/3})^3 = 8x^2, \quad (4x^{-1/3})^2 = 4^2 \cdot (x^{-1/3})^2 = 16x^{-2/3}$$

Step 2: Combine terms

$$\frac{8x^2}{16x^{-2/3}} = \frac{8}{16} \cdot \frac{x^2}{x^{-2/3}} = \frac{1}{2} \cdot x^{2+2/3} = \frac{1}{2}x^{\frac{8}{3}}$$

Final result:

The Number e

Consider the example of compound interest in a savings account with this form:

$$C(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

where:

- P is the principal (initial investment),
- r is the annual interest rate,
- t is the number of years,
- n is the number of times the interest is compounded per year.

What happens as $n \to \infty$? Let us consider $m = n/r$, the formula becomes:

$$C(t) = P\left(1 + \frac{1}{m}\right)^{mrt} = PA^{rt}$$

where

$$A = \left(1 + \frac{1}{m}\right)^{m}$$

Values of $\left(1+\frac{1}{m}\right)^m$ as $m \to \infty$

m	10	100	1,000	10,000	100,000	1,000,000
$\left(1+\frac{1}{m}\right)^m$	2.5937	2.7048	2.71692	2.71815	2.718268	2.718280

Looking at this table, it appears that $\left(1+\frac{1}{m}\right)^m$ is approaching a number between 2.7 and 2.8 as $m \to \infty$. In fact, $\left(1+\frac{1}{m}\right)^m$ does approach a specific number as $m \to \infty$, which we call e. To six decimal places of accuracy,

$$e \approx 2.718282.$$

Savings Account Example

Then:

$$A(t) = Pe^{rt}.$$

This function may be familiar. Since functions involving base e arise often in applications, we call the function $f(x) = e^x$ the **natural exponential function**. The letter e was first used by **Euler** during the 1720s.

Exponential Growth Example: Population

Suppose a population of 500 is growing at a continuous annual growth rate of $r = 5.5\%$.

Let t denote the number of years after the initial population count and $P(t)$ denote the population at time t.

- Find a formula for $P(t)$, the population at any time t.
- Find the population after 10 years and after 20 years.

Logarithmic Functions

Using our understanding of exponential functions, we can discuss their inverses, which are the **logarithmic functions**.

These are useful when dealing with phenomena that vary over a wide range of values, such as pH in chemistry or decibels in sound levels.

The exponential function $f(x) = b^x$ is one-to-one, with domain $(-\infty, \infty)$ and range $(0, \infty)$, so it has an inverse called the **logarithmic function** with base b, denoted $\log_b$.

$$\log_b(x) = y \quad \text{if and only if} \quad b^y = x.$$

For example:

$$\log_2(8) = 3 \quad \text{since} \quad 2^3 = 8, \quad \log_{10}\left(\frac{1}{100}\right) = -2 \quad \text{since} \quad 10^{-2} = \frac{1}{100}.$$

The most commonly used logarithmic function is the natural logarithm, $\ln(x)$, where $\log_e(x) = \ln(x)$. For example:

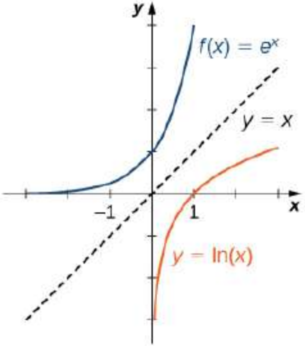
$$\ln(e) = 1, \quad \ln(e^3) = 3, \quad \ln(1) = 0.$$

Inverse Relationship

Since the functions $f(x) = e^x$ and $g(x) = \ln(x)$ are inverses of each other:

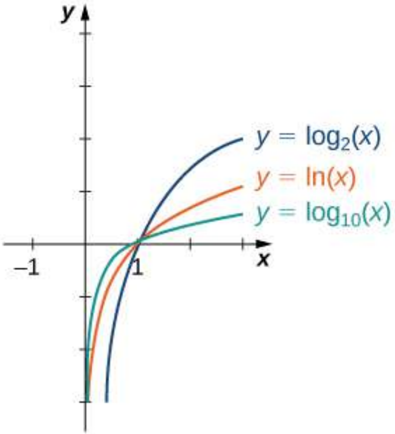
$$g(f(x)) = \ln(e^x) = x \quad \text{and} \quad f(g(x)) = e^{\ln(x)} = x.$$

These identities hold for all x where the logarithm is defined, demonstrating the symmetry and inverse nature of the functions.



Graph of Logarithmic Functions

In general, for any base $b > 0$, $b \neq 1$, the function $g(x) = \log_b(x)$ is symmetric about the line $y = x$ with the function $f(x) = b^x$. Using this fact and the graphs of exponential functions, we can graph functions $\log_b(x)$ for several values of $b > 1$.



Properties of Logarithms

If $a, b, c > 0$, $b \neq 1$, and r is any real number, then the following properties of logarithms hold:

1. **Product Property:**

$$\log_b(ac) = \log_b(a) + \log_b(c)$$

2. **Quotient Property:**

$$\log_b\left(\frac{a}{c}\right) = \log_b(a) - \log_b(c)$$

3. **Power Property:**

$$\log_b(a^r) = r\log_b(a)$$

These properties are useful when simplifying logarithmic expressions.

Solving Equations Involving Exponential Functions

Solve each of the following equations for x:

1. $5^x = 2$
2. $e^x + 6e^{-x} = 5$

Use logarithmic properties or algebraic manipulation to isolate x in each case.

Solving Equations Involving Logarithmic Functions

Solve each of the following equations for x:

1. $\ln\left(\frac{1}{x}\right) = 4$
2. $\log_{10}(\sqrt{x}) + \log_{10}(x) = 2$
3. $\ln(2x) - 3\ln(x^2) = 0$

Use logarithmic properties and algebraic manipulation to solve for x in each equation.

Rule: Change-of-Base Formulas

Let $a > 0$, $b > 0$, and $a \neq 1$, $b \neq 1$.

- $a^x = b^{x \log_b a}$ for any real number x.
- If $b = e$, this equation reduces to:

$$a^x = e^{x \log_e a} = e^{x \ln a}$$

- The change-of-base formula for logarithms:

$$\log_a x = \frac{\log_b x}{\log_b a}$$

  for any real number $x > 0$.
- If $b = e$, this equation reduces to:

$$\log_a x = \frac{\ln x}{\ln a}$$

Solve each of the following equations for x:

1. $e^x + 3e^{-x} = 4$
2. $2\log(x) + \log(2x) = 3$
3. $5^x = 125$
4. $\ln(x) - \ln(3) = 2$
5. $4e^{2x} = 12$

Use algebraic manipulation, properties of logarithms, and exponential rules to isolate x in each case.

Find the Inverses of the Following Functions

Solve for the inverse of the following functions:

- $f(x) = 2^x$
- $f(x) = e^{2x}$
- $f(x) = \ln(x+1)$
- $f(x) = \ln(x-2)$
- $f(x) = \ln(3x-1)$

Find the inverse for each function by applying logarithmic or exponential rules.

Limits

Clotilde Djuikem

Introduction to Limits

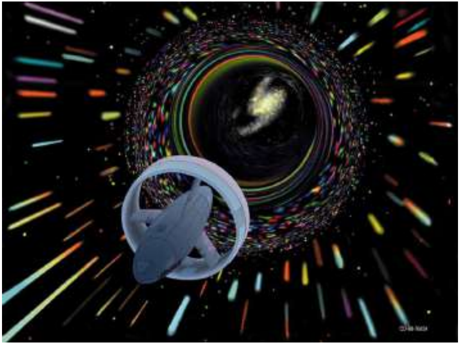


Figure: The vision of human exploration by NASA to distant parts of the universe illustrates the idea of space travel at high speeds. But, is there a limit to how fast a spacecraft can go? (credit: NASA)

Introduction: Science fiction writers often imagine spaceships that can travel to far-off planets in distant galaxies. However, in reality, there are physical limits that govern how fast an object, such as a spacecraft, can move.

Concept

In mathematics, the concept of a *limit* helps us understand behavior as values approach a certain point. This course will explore the fundamental idea of limits, a key concept in calculus.

Learning Objectives

By the end of this course, students will be able to:

- Using correct notation, describe the limit of a function.
- Use a table of values to estimate the limit of a function or to identify when the limit does not exist.
- Use a graph to estimate the limit of a function or to identify when the limit does not exist.
- Define one-sided limits and provide examples.
- Explain the relationship between one-sided and two-sided limits.
- Using correct notation, describe an infinite limit.
- Define a vertical asymptote.

Definition of a Limit

Definition:

Let $f(x)$ be a function defined at all values in an open interval containing a, with the possible exception of a itself, and let L be a real number.
If all values of the function $f(x)$ approach the real number L as the values of x (where $x \neq a$) approach the number a, then we say that the limit of $f(x)$ as x approaches a is L. Symbolically:

$$\lim_{x \to a} f(x) = L$$

Example: Consider the function $f(x) = \frac{x^2-1}{x-1}$.
Simplifying the expression:

$$f(x) = \frac{(x-1)(x+1)}{x-1} = x+1, \quad \text{for} \quad x \neq 1$$

So:

$$\lim_{x \to 1} \frac{x^2-1}{x-1} = \lim_{x \to 1} (x+1) = 2$$

Problem-Solving Strategy: Evaluating a Limit Using a Table of Functional Values

To evaluate $\lim_{x \to a} f(x)$, follow these steps:

1. Complete a table of functional values by choosing two sets of x-values:
   - One set of values approaching a from the left ($x < a$).
   - Another set of values approaching a from the right ($x > a$).
2. Example table:

x	$f(x)$	x	$f(x)$
$a - 0.1$	$f(a - 0.1)$	$a + 0.1$	$f(a + 0.1)$
$a - 0.01$	$f(a - 0.01)$	$a + 0.01$	$f(a + 0.01)$
$a - 0.001$	$f(a - 0.001)$	$a + 0.001$	$f(a + 0.001)$
$a - 0.0001$	$f(a - 0.0001)$	$a + 0.0001$	$f(a + 0.0001)$

3. Examine the values in each column to determine if both columns approach a common value L. If so, $\lim_{x \to a} f(x) = L$.
4. Alternatively, use a graphing calculator or computer software to graph $f(x)$. Use the trace feature to observe the behavior of y-values as x-values approach a.

Evaluating a Limit Using a Table of Functional Values

Problem: Evaluate $\lim_{x\to 0} \frac{\sin(x)}{x}$ using a table of functional values.
Solution:
We have calculated the values of $f(x) = \frac{\sin(x)}{x}$ for the values of x listed below.

x	$\frac{\sin(x)}{x}$	x	$\frac{\sin(x)}{x}$
-0.1	0.998334	0.1	0.998334
-0.01	0.999983	0.01	0.999983
-0.001	0.9999998	0.001	0.9999998
-0.0001	0.9999999998	0.0001	0.9999999998

As we observe the values in both columns approaching 1, it is reasonable to conclude:

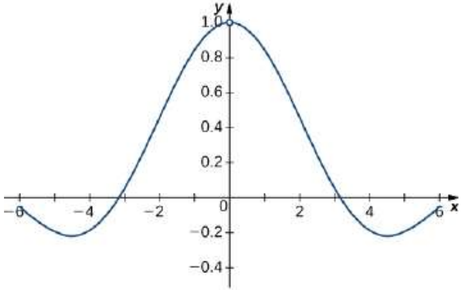
$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1$$

Conclusion: Evaluating the Limit $\lim_{x\to 0} \frac{\sin(x)}{x}$

As we read down each $\frac{\sin(x)}{x}$ column, we see that the values in each column approach 1. Thus, it is reasonable to conclude that:

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1$$

A calculator or computer-generated graph of $f(x) = \frac{\sin(x)}{x}$ would be similar to the graph shown below, confirming our estimate.



Two Important Limits

Let a be a real number and c be a constant. Then:

$$\lim_{x \to a} x = a$$

This limit tells us that as x approaches a, the function $f(x) = x$ approaches a.

$$\lim_{x \to a} c = c$$

This limit tells us that a constant function always has the same limit, regardless of x.

The Existence of a Limit

As we consider the limit in the next example, remember that for the limit of a function to exist at a point, the functional values must approach a single real-number value. If they do not approach a single value, the limit does not exist.

Evaluating a Limit That Fails to Exist: $\lim_{x\to 0} \sin\left(\frac{1}{x}\right)$ using a table of values:

x	$\sin\left(\frac{1}{x}\right)$	x	$\sin\left(\frac{1}{x}\right)$
-0.1	0.5440	0.1	-0.5440
-0.01	0.5064	0.01	-0.5064
-0.001	-0.8269	0.001	0.8269
-0.0001	0.3056	0.0001	-0.3056

The y-values do not approach a single value, indicating that:

$$\lim_{x\to 0} \sin\left(\frac{1}{x}\right) \quad \text{does not exist (DNE).}$$

The graph of $\sin\left(\frac{1}{x}\right)$ shows that as $x \to 0$, the function oscillates between -1 and 1.

One-Sided Limits

Understanding One-Sided Limits

Sometimes, indicating that the limit of a function does not exist at a point does not provide enough information about the behavior of the function at that point. To address this, we introduce the idea of one-sided limits.

Consider the function $g(x) = \frac{|x-2|}{x-2}$. As we pick values of x close to 2, $g(x)$ does not approach a single value, meaning the limit as $x \to 2$ does not exist.

However, we can analyze one-sided limits:

$$\lim_{x\to 2^-} g(x) = -1 \quad \text{and} \quad \lim_{x\to 2^+} g(x) = 1$$

Definition: One-Sided Limits

Limit from the Left

Let $f(x)$ be a function defined at all values in an open interval of the form (b, a), and let L be a real number. If the values of $f(x)$ approach L as $x \to a^-$, then:

$$\lim_{x \to a^-} f(x) = L$$

Limit from the Right

Let $f(x)$ be a function defined at all values in an open interval of the form (a, c), and let L be a real number. If the values of $f(x)$ approach L as $x \to a^+$, then:

$$\lim_{x \to a^+} f(x) = L$$

Example with Table of Values

For the function:

$$f(x) = \begin{cases} x+1 & \text{if } x < 2 \\ x^2 - 4 & \text{if } x \geq 2 \end{cases}$$

evaluate each of the following limits: $\lim_{x \to 2^-} f(x)$ and $\lim_{x \to 2^+} f(x)$

Table of Functional Values

x	$f(x) = x + 1$	x	$f(x) = x^2 - 4$
1.9	2.9	2.1	0.41
1.99	2.99	2.01	0.0401
1.999	2.999	2.001	0.004001
1.9999	2.9999	2.0001	0.00040001

For values approaching $x = 2$ from the left and right, we observe:

$$\lim_{x \to 2^-} f(x) = 3 \quad \text{and} \quad \lim_{x \to 2^+} f(x) = 0$$

Understanding Infinite Limits

Evaluating the limit of a function at a point or from the right and left helps us characterize the behavior of a function around a given value. However, some functions do not have finite limits.
Let $h(x) = \frac{1}{(x-2)^2}$. From the graph, as x approaches 2, the values of $h(x)$ become infinitely large. In this case, we say that the limit of $h(x)$ as x approaches 2 is positive infinity.

$$\lim_{x \to 2} h(x) = +\infty$$

Definition: Infinite Limits

We define three types of infinite limits:

- **Infinite limits from the left:** Let $f(x)$ be defined at all values in an open interval of the form (b, a). If the values of $f(x)$ increase without bound as x approaches a from the left, then:

$$\lim_{x \to a^-} f(x) = +\infty$$

Evaluating Limits of $\frac{1}{x}$

Evaluate each of the following limits, if possible. Use a table of functional values and a graph of $f(x) = \frac{1}{x}$ to confirm your conclusion.

$$\lim_{x \to 0^-} \frac{1}{x} \qquad \lim_{x \to 0^+} \frac{1}{x} \qquad \lim_{x \to 0} \frac{1}{x}$$

Solution: Begin by constructing a table of functional values.

x	$\frac{1}{x}$	x	$\frac{1}{x}$
-0.1	-10	0.1	10
-0.01	-100	0.01	100
-0.001	-1000	0.001	1000
-0.0001	-10,000	0.0001	10,000
-0.00001	-100,000	0.00001	100,000
-0.000001	-1,000,000	0.000001	1,000,000

Conclusion: $\lim_{x \to 0^-} \frac{1}{x} = -\infty$ and $\lim_{x \to 0^+} \frac{1}{x} = +\infty$
Since the left and right limits are not equal, we conclude:

$$\lim_{x \to 0} \frac{1}{x} \quad \text{does not exist (DNE).}$$

Infinite Limits from Positive Integers

If n is a positive even integer, then:

$$\lim_{x \to a} \frac{1}{(x-a)^n} = +\infty$$

If n is a positive odd integer, then:

$$\lim_{x \to a^+} \frac{1}{(x-a)^n} = +\infty \quad \text{and} \quad \lim_{x \to a^-} \frac{1}{(x-a)^n} = -\infty$$

Points on the graph of $f(x) = \frac{1}{(x-a)^n}$ near a are very close to the vertical line $x = a$, which we call a vertical asymptote.

Definition: Vertical Asymptote

Definition

Let $f(x)$ be a function. The line $x = a$ is a vertical asymptote if any of the following hold:

$$\lim_{x \to a^-} f(x) = +\infty \quad \text{or} \quad \lim_{x \to a^-} f(x) = -\infty$$

$$\lim_{x \to a^+} f(x) = +\infty \quad \text{or} \quad \lim_{x \to a^+} f(x) = -\infty$$

$$\lim_{x \to a} f(x) = +\infty \quad \text{or} \quad \lim_{x \to a} f(x) = -\infty$$

Key Concepts

- A table of values or graph may be used to estimate a limit.
- If the limit of a function at a point does not exist, it is still possible that the limits from the left and right at that point may exist.
- If the limits of a function from the left and right exist and are equal, then the limit of the function is that common value.
- We may use limits to describe infinite behavior of a function at a point.

Key Equations

- **Intuitive Definition of the Limit** $\lim_{x\to a} f(x) = L$
- **Two Important Limits** $\lim_{x\to a} x = a$ and $\lim_{x\to a} c = c$
- **One-Sided Limits**

$$\lim_{x\to a^-} f(x) = L \quad \text{and} \quad \lim_{x\to a^+} f(x) = L$$

- **Infinite Limits from the Left**

$$\lim_{x\to a^-} f(x) = +\infty \quad \text{or} \quad \lim_{x\to a^-} f(x) = -\infty$$

- **Infinite Limits from the Right**

$$\lim_{x\to a^+} f(x) = +\infty \quad \text{or} \quad \lim_{x\to a^+} f(x) = -\infty$$

- **Two-Sided Infinite Limits**

$$\lim_{x\to a} f(x) = +\infty \quad \text{if} \quad \lim_{x\to a^-} f(x) = +\infty \quad \text{and} \quad \lim_{x\to a^+} f(x) = +\infty$$

$$\lim_{x\to a} f(x) = -\infty \quad \text{if} \quad \lim_{x\to a^-} f(x) = -\infty \quad \text{and} \quad \lim_{x\to a^+} f(x) = -\infty$$

Examples of Limits to Compute

1. **Basic Limits:**

$$\lim_{x \to 3} (2x + 1), \quad \lim_{x \to -1} (x^2 + 4x + 3), \quad \lim_{x \to 0} \frac{x^2 - 4}{x - 2}$$

2. **One-Sided Limits:**

$$\lim_{x \to 1^-} \frac{1}{x - 1}, \quad \lim_{x \to 1^+} \frac{1}{x - 1}$$

$$\lim_{x \to 0^-} \frac{|x|}{x}, \quad \lim_{x \to 0^+} \frac{|x|}{x}$$

Examples of Limits to Compute (Cont.)

3. Infinite Limits:

$$\lim_{x \to 2^-} \frac{1}{(x-2)}, \quad \lim_{x \to 2^+} \frac{1}{(x-2)}$$

$$\lim_{x \to 0} \frac{1}{x^2}$$

4. Trigonometric Limits:

$$\lim_{x \to 0} \frac{\sin(x)}{x}, \quad \lim_{x \to 0} \frac{1-\cos(x)}{x^2}, \quad \lim_{x \to \pi/2^-} \tan(x)$$

$$\lim_{x \to 0} \sin(2x)$$

Examples of Limits to Compute (Cont.)

5. Exponential and Logarithmic Limits:

$$\lim_{x \to 0^+} \ln(x), \quad \lim_{x \to \infty} \frac{1}{e^x}, \quad \lim_{x \to \infty} xe^{-x}$$

6. Special Limits:

$$\lim_{x \to \infty} \frac{3x^2 + 2x}{x^2 - 4x + 1}, \quad \lim_{x \to 1} \frac{x^2 - 1}{x - 1}, \quad \lim_{x \to 0} \frac{\ln(1 + x)}{x}$$

Exercise Worksheet: Finding the Domain of Functions and Slope

Clotilde Djuikem

Instructions

For each of the following functions, determine the domain. Show your work and provide an explanation for your answer.

Exercise 1:

$$f(x) = 2x + 5$$

Domain:

Exercise 2:

$$f(x) = \frac{1}{x - 3}$$

Domain:

Exercise 3:

$$f(x) = \sqrt{x + 4}$$

Domain:

Exercise 4:

$$f(x) = \frac{\sqrt{x - 1}}{x + 2}$$

Domain:

Exercise 5:

$$f(x) = \frac{x^2 - 1}{x^2 - 4}$$

Domain:

Bonus Question

Find the domain of the composite function:

$$f(g(x)) = \frac{1}{\sqrt{x-2}} \quad \text{where} \quad g(x) = \sqrt{x-1}$$

Domain:

Slope Example: Finding the Slope Between Two Points

The slope between two points (x_1, y_1) and (x_2, y_2) is calculated using the formula:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Example: Find the slope of the line passing through the points $(2, 3)$ and $(5, 7)$.
Solution:

$$m = \frac{7-3}{5-2} = \frac{4}{3}$$

The slope of the line is $\frac{4}{3}$.

Now, try finding the slope for the following points:
Exercise: Find the slope between the points $(-1, -2)$ and $(3, 4)$.
Slope:

Instructions

For each of the following functions, evaluate the limit. Show your work and provide an explanation for your answer.

Exercise 1:

$$\lim_{x \to 3} (2x + 1)$$

Solution:

Exercise 2:

$$\lim_{x \to 0} \frac{x^2 - 4}{x - 2}$$

Solution:

Exercise 3:

$$\lim_{x \to 1^-} \frac{1}{x-1}$$

Solution:

Exercise 4:

$$\lim_{x \to 0^+} \frac{|x|}{x}$$

Solution:

Exercise 5:

$$\lim_{x \to 2^+} \frac{1}{(x-2)}$$

Solution:

Exercise 6:

$$\lim_{x \to 0} \frac{\sin(x)}{x}$$

Solution:

Bonus Question

Evaluate the following limit:

$$\lim_{x \to \infty} \frac{3x^2 + 2x}{x^2 - 4x + 1}$$

Solution:

Trigonometric Limit Example:

Example: Evaluate the limit:

$$\lim_{x \to 0} \frac{1 - \cos(x)}{x^2}$$

Solution: