# The Limit laws

Clotilde Djuikem

# Learning Objectives

1. Recognize the basic limit laws.
2. Use the limit laws to evaluate the limit of a function.
3. Evaluate the limit of a function by factoring.
4. Use the limit laws to evaluate the limit of a polynomial or rational function.
5. Evaluate the limit of a function by factoring or by using conjugates.
6. Evaluate the limit of a function by using the squeeze theorem.

# Basic Limit Results

## The first two limit laws

For any real number $a$ and any constant $c$:

$$\lim_{x \to a} x = a$$

$$\lim_{x \to a} c = c$$

## Examples

1. $\lim_{x \to 2} x = 2$
2. $\lim_{x \to 5} 3 = 3$
3. $\lim_{x \to 0} (-7) = -7$
4. $\lim_{x \to -4} x^2 = 16$
5. $\lim_{x \to 1} (2x + 1) = 3$

# Limit Laws (Part 1)

Let $f(x)$ and $g(x)$ be defined for all $x \neq a$ over some open interval containing $a$. Assume that $L$ and $M$ are real numbers such that

$$\lim_{x \to a} f(x) = L \text{ and } \lim_{x \to a} g(x) = M$$

Let $c$ be a constant. Then, each of the following statements holds:

**Sum Law**

$$\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = L + M$$

**Difference Law**

$$\lim_{x \to a} (f(x) - g(x)) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x) = L - M$$

**Constant Multiple Law**

$$\lim_{x \to a} (c \cdot f(x)) = c \cdot \lim_{x \to a} f(x) = c \cdot L$$

# Limit Laws (Part 2)

## Product Law

$$\lim_{x \to a} (f(x) \cdot g(x)) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) = L \cdot M$$

## Quotient Law

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{L}{M}, \quad \text{for } M \neq 0$$

## Power Law

$$\lim_{x \to a} (f(x))^n = \left( \lim_{x \to a} f(x) \right)^n = L^n$$

## Root Law

$$\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)} = \sqrt[n]{L}$$

For all $L$ if $n$ is odd, and for $L \geq 0$ if $n$ is even.

# Evaluating a Limit Using Limit Laws (Example 1)

**Use the limit laws to evaluate**

$$\lim_{x \to -3} (4x + 2).$$

**Solution:**

$$\begin{aligned}
\lim_{x \to -3} (4x + 2) &= \lim_{x \to -3} 4x + \lim_{x \to -3} 2 \\
\text{(Apply the Sum Law)} & \\
&= 4 \cdot \lim_{x \to -3} x + \lim_{x \to -3} 2 \\
\text{(Apply the Constant Multiple Law)} & \\
&= 4 \cdot (-3) + 2 \\
\text{(Substitute } x = -3\text{)} & \\
&= -12 + 2 \\
&= -10
\end{aligned}$$

# Evaluating a Limit Using Limit Laws (Example 2)

**Use the limit laws to evaluate**

$$\lim_{x \to 2} \frac{2x^2 - 3x + 1}{x^3 + 4}.$$

**Solution:**

$$\lim_{x \to 2} \frac{2x^2 - 3x + 1}{x^3 + 4} = \frac{\lim_{x \to 2}(2x^2 - 3x + 1)}{\lim_{x \to 2}(x^3 + 4)}$$

(Apply the Quotient Law)

$$= \frac{2 \cdot \lim_{x \to 2} x^2 - 3 \cdot \lim_{x \to 2} x + \lim_{x \to 2} 1}{(\lim_{x \to 2} x)^3 + \lim_{x \to 2} 4}$$

(Apply the Sum Law and Constant Multiple Law)

$$= \frac{2 \cdot (2)^2 - 3 \cdot 2 + 1}{2^3 + 4}$$

(Substitute $x = 2$)

$$= \frac{2 \cdot 4 - 6 + 1}{8 + 4} = \frac{8 - 6 + 1}{12} = \frac{3}{12} = \frac{1}{4}$$

# Evaluating a Limit Using Limit Laws (Example 3)

**Use the limit laws to evaluate**

$$\lim_{x \to 4} \sqrt{x^2 + 1}.$$

**Solution:**

$$\begin{aligned}
\lim_{x \to 4} \sqrt{x^2 + 1} &= \sqrt{\lim_{x \to 4} (x^2 + 1)} \\
\text{(Apply the Root Law)} & \\
&= \sqrt{(4)^2 + 1} \\
\text{(Substitute } x = 4\text{)} & \\
&= \sqrt{16 + 1} \\
&= \sqrt{17} \\
&= \sqrt{17}
\end{aligned}$$

# Evaluating a Limit Using Limit Laws (Example 4)

**Use the limit laws to evaluate**

$$\lim_{x \to 2} (x^2 \cdot \sin(x)).$$

**Solution:**

$$\begin{aligned}
\lim_{x \to 2} (x^2 \cdot \sin(x)) &= \left(\lim_{x \to 2} x^2\right) \cdot \left(\lim_{x \to 2} \sin(x)\right) \\
\text{(Apply the Product Law)} & \\
&= (2)^2 \cdot \sin(2) \\
\text{(Substitute } x = 2\text{)} & \\
&= 4 \cdot \sin(2) \\
&= 4 \sin(2)
\end{aligned}$$

**Use the limit laws to evaluate**

$$\lim_{x\to 6} \frac{2x-1}{\sqrt[3]{x}+4}.$$

**Solution:**

$$\lim_{x\to 6} \frac{2x-1}{\sqrt[3]{x}+4} = \frac{\lim_{x\to 6}(2x-1)}{\lim_{x\to 6}(\sqrt[3]{x}+4)}$$

(Apply the Quotient Law)

$$= \frac{\lim_{x\to 6}(2x-1)}{\lim_{x\to 6}\sqrt[3]{x} + \lim_{x\to 6} 4}$$

(Apply the Sum Law)

$$= \frac{2\cdot \lim_{x\to 6} x - 1}{\sqrt[3]{\lim_{x\to 6} x} + 4}$$

(Apply the Constant Multiple Law and Power Law)

$$= \frac{2\cdot 6-1}{\sqrt[3]{6}+4} = \frac{12-1}{\sqrt[3]{6}+4} = \frac{11}{\sqrt[3]{6}+4}$$

# Limits of Polynomial and Rational Functions

## Limits of Polynomial Functions

Let $p(x)$ and $q(x)$ be polynomial functions. Let $a$ be a real number. Then,

$$\lim_{x\to a} p(x) = p(a)$$

$$\lim_{x\to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)} \quad \text{when } q(a) \neq 0.$$

## Example: **Evaluate**

$$\lim_{x\to 3} \frac{2x^2 - 3x + 1}{5x + 4}.$$

**Solution:** Since 3 is in the domain of that rational function we can calculate the limit by substituting $x = 3$ into the function. Thus,

$$\lim_{x\to 3} \frac{2x^2 - 3x + 1}{5x + 4} = \frac{2(3)^2 - 3(3) + 1}{5(3) + 4} = \frac{2 \cdot 9 - 9 + 1}{15 + 4} = \frac{18 - 9 + 1}{19} = \frac{10}{19}.$$

# Evaluating a Limit of a Rational Function (Example 1)

## Evaluate

$$\lim_{x \to 2} \frac{3x^2 - 4x + 1}{x + 1}.$$

## Solution

Since $x = 2$ is in the domain of the function

$$f(x) = \frac{3x^2 - 4x + 1}{x + 1},$$

we can calculate the limit by direct substitution:

$$\begin{aligned} \lim_{x \to 2} \frac{3x^2 - 4x + 1}{x + 1} &= \frac{3(2)^2 - 4(2) + 1}{2 + 1} \\ &= \frac{12 - 8 + 1}{3} = \frac{5}{3}. \end{aligned}$$

# Evaluating a Limit of a Rational Function (Example 2)

## Evaluate

$$\lim_{x \to 4} \frac{x^2 - 16}{x - 4}.$$

## Solution

Since $x = 4$ is in the domain of the function

$$f(x) = \frac{x^2 - 16}{x - 4},$$

we can use factoring:

$$\lim_{x \to 4} \frac{x^2 - 16}{x - 4} = \lim_{x \to 4} \frac{(x - 4)(x + 4)}{x - 4} = \lim_{x \to 4} (x + 4).$$

Substituting $x = 4$:

$$4 + 4 = 8.$$

# Steps to Solve Limits with Indeterminate Form $\frac{0}{0}$

## Step 1: Verify the Indeterminate Form

- Ensure that the function has the form $\frac{f(x)}{g(x)} = \frac{0}{0}$ and cannot be evaluated directly using limit laws.

## Step 2: Simplify the Expression

- Try to find a function $h(x) = \frac{f(x)}{g(x)}$ for all $x \neq a$ near $a$. - Factor and cancel common terms if $f(x)$ and $g(x)$ are polynomials. - If square roots are involved, multiply by the conjugate. - If the fraction is complex, simplify it first.

## Step 3: Apply Limit Laws

- After simplifying, apply the appropriate limit laws to calculate the final limit.

# Example: Calculating $\lim_{x \to 2} \frac{x^2-4}{x-2}$

## Step 1: Verify the Indeterminate Form

Substitute $x = 2$ into the function:

$$\frac{(2)^2 - 4}{2 - 2} = \frac{4 - 4}{0} = \frac{0}{0}$$

This results in the indeterminate form $\frac{0}{0}$.

## Step 2: Simplify the Expression

Factor the numerator $x^2 - 4$ (difference of squares):

$$\frac{x^2 - 4}{x - 2} = \frac{(x - 2)(x + 2)}{x - 2} = x + 2 \quad \text{for} \quad x \neq 2$$

## Step 3: Apply the Limit Laws

Now substitute $x = 2$ into the simplified expression:

$$\lim_{x \to 2}(x + 2) = 2 + 2 = 4$$

# Example: Evaluating $\lim_{x \to 5} \frac{\sqrt{x-1}-2}{x-5}$

## Step 1: Verify the Indeterminate Form

Substitute $x = 5$ into the expression:

$$\frac{\sqrt{5-1}-2}{5-5} = \frac{\sqrt{4}-2}{0} = \frac{2-2}{0} = \frac{0}{0}$$

This gives the indeterminate form $\frac{0}{0}$, so we proceed to simplify.

## Step 2: Simplify Using Conjugates

Multiply the numerator and denominator by the conjugate of the numerator:

$$\frac{\sqrt{x-1}-2}{x-5} \cdot \frac{\sqrt{x-1}+2}{\sqrt{x-1}+2} = \frac{(\sqrt{x-1})^2 - 2^2}{(x-5)(\sqrt{x-1}+2)}$$

Simplify the numerator:

$$= \frac{x-1-4}{(x-5)(\sqrt{x-1}+2)} = \frac{x-5}{(x-5)(\sqrt{x-1}+2)} = \frac{1}{\sqrt{x-1}+2}$$

## Step 3: Apply the Limit Laws

Now substitute $x = 5$ into the simplified expression:

$$\lim_{x \to 5} \frac{1}{\sqrt{x-1}+2} = \frac{1}{\sqrt{5-1}+2} = \frac{1}{2+2} = \frac{1}{4}$$

# Example: Evaluating $\lim_{x\to1} \frac{\frac{1}{x+1}-\frac{1}{2}}{x-1}$

## Step 1: Verify the Indeterminate Form

Substitute $x = 1$ into the expression:

$$\frac{\frac{1}{1+1}-\frac{1}{2}}{1-1} = \frac{\frac{1}{2}-\frac{1}{2}}{0} = \frac{0}{0}$$

This gives the indeterminate form $\frac{0}{0}$, so we proceed to simplify.

## Step 2: Simplify the Complex Fraction

Simplify the numerator by combining the two fractions:

$$\frac{1}{x+1} - \frac{1}{2} = \frac{2-(x+1)}{2(x+1)} = \frac{1-x}{2(x+1)}$$

Substitute this into the limit expression:

$$\lim_{x\to1} \frac{\frac{1-x}{2(x+1)}}{x-1} = \lim_{x\to1} \frac{1-x}{2(x+1)} \cdot \frac{1}{x-1} = \lim_{x\to1} \frac{-(x-1)}{2(x+1)(x-1)} = \lim_{x\to1} \frac{-1}{2(x+1)}$$

## Step 3: Apply the Limit Laws

Now substitute $x = 1$:

$$\frac{-1}{2(1+1)} = -\frac{1}{4}$$

# Evaluating a Limit When the Limit Laws Do Not Apply

## Problem

Evaluate $\lim_{x\to 0}\left(\frac{1}{x}+\frac{5}{x(x-5)}\right)$.

# Evaluating a Limit When the Limit Laws Do Not Apply

## Problem

Evaluate $\lim_{x\to 0}\left(\frac{1}{x}+\frac{5}{x(x-5)}\right)$.

## Solution

Both $\frac{1}{x}$ and $\frac{5}{x(x-5)}$ fail to have a limit at zero. Since neither of the two functions has a limit at zero, we cannot apply the sum law for limits; we must use a different strategy. In this case, we find the limit by performing addition and then applying one of our previous strategies.

Observe that:

$$\frac{1}{x}+\frac{5}{x(x-5)}=\frac{x-5+5}{x(x-5)}=\frac{x}{x(x-5)}$$

Thus,

$$\lim_{x\to 0}\left(\frac{1}{x}+\frac{5}{x(x-5)}\right)=\lim_{x\to 0}\frac{x}{x(x-5)}=\lim_{x\to 0}\frac{1}{x-5}=-\frac{1}{5}$$

Therefore,

$$\lim_{x\to 0}\left(\frac{1}{x}+\frac{5}{x(x-5)}\right)=-\frac{1}{5}.$$

# Indeterminate Forms in Limits

## Common Indeterminate Forms

When evaluating limits, certain expressions are indeterminate, meaning they require further analysis to find the limit. Here are the most common indeterminate forms:

- $\frac{0}{0}$ - Example: $\lim_{x\to 0} \frac{\sin x}{x}$
- $\frac{\infty}{\infty}$ - Example: $\lim_{x\to\infty} \frac{x^2}{e^x}$
- $0 \cdot \infty$ - Example: $\lim_{x\to 0} x \cdot \ln x$
- $\infty - \infty$ - Example: $\lim_{x\to\infty}(\sqrt{x^2+1} - x)$
- $1^\infty$ - Example: $\lim_{x\to 0^+} (1+x)^{1/x}$
- $0^0$ - Example: $\lim_{x\to 0^+} x^x$
- $\infty^0$ - Example: $\lim_{x\to 0^+}(x^{-1})^x$

These indeterminate forms require techniques such as L'Hopital's Rule, factoring, or algebraic manipulation to resolve.

# Evaluating a Limit of the Form $\frac{K}{0}, K \neq 0$ Using the Limit Laws

## Problem

Evaluate $\lim_{x\to 2} \frac{x-3}{x^2-2x}$.

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## Problem

Evaluate $\lim_{x\to 2} \frac{x-3}{x^2-2x}$.

## Solution

**Step 1.** After substituting $x = 2$, we see that this limit has the form $\frac{-1}{0}$. That is, as $x$ approaches 2 from the left, the numerator approaches $-1$ and the denominator approaches 0. Consequently, the magnitude of $\frac{x-3}{x(x-2)}$ becomes infinite. To get a better idea of what the limit is, we need to factor the denominator:

$$\lim_{x\to 2} \frac{x-3}{x^2-2x} = \lim_{x\to 2} \frac{x-3}{x(x-2)}.$$

**Step 2.** Since $x - 2$ is the only part of the denominator that is zero when 2 is substituted, we then separate $\frac{1}{x-2}$ from the rest of the function:

$$= \lim_{x\to 2} \frac{x-3}{x} \cdot \frac{1}{x-2}.$$

**Step 3.**

$$\lim_{x\to 2} \frac{x-3}{x} = \frac{-1}{2} \quad \text{and} \quad \lim_{x\to 2} \frac{1}{x-2} = -\infty.$$

Therefore, the product of $\frac{x-3}{x}$ and $\frac{1}{x-2}$ has a limit of $+\infty$:

# The Squeeze Theorem

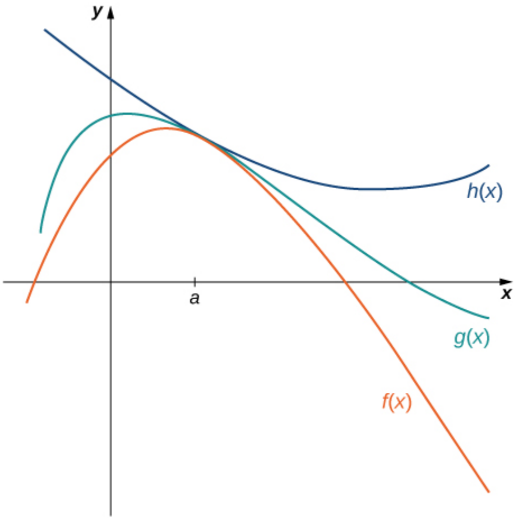


Figure: The Squeeze Theorem

# The Squeeze Theorem

## The Squeeze Theorem

Let $f(x)$, $g(x)$, and $h(x)$ be defined for all $x \neq a$ such that:

$$f(x) \leq g(x) \leq h(x)$$

for all $x \neq a$ in an open interval containing $a$ and

$$\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x)$$

where $L$ is a real number. Then:

$$\lim_{x \to a} g(x) = L$$

# Applying the Squeeze Theorem

**Problem:**

Apply the Squeeze Theorem to evaluate $\lim_{x \to 0} x \cos x$.

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**Problem:**

Apply the Squeeze Theorem to evaluate $\lim_{x\to 0} x \cos x$.

**Solution:**

We know that for all $x$,

$$-1 \leq \cos x \leq 1$$

Multiplying through by $x$ (assuming $x \geq 0$) gives:

$$-x \leq x \cos x \leq x$$

By taking the limit as $x \to 0$ on both sides:

$$\lim_{x\to 0} -x = 0 \quad \text{and} \quad \lim_{x\to 0} x = 0$$

Thus, by the Squeeze Theorem:

$$\lim_{x\to 0} x \cos x = 0$$

# Evaluating a Limit of a Rational Function (Example 3)

## Evaluate

$$\lim_{x \to 0} \frac{\sin x}{x}.$$

## Solution

This is a standard limit result that is known to be:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

# Evaluating an Important Trigonometric Limit

**Problem:**

Evaluate $\lim_{\theta\to 0} \frac{1-\cos\theta}{\theta}$.

# Evaluating an Important Trigonometric Limit

## Problem:

Evaluate $\lim_{\theta\to 0} \frac{1-\cos\theta}{\theta}$.

## Solution

In the first step, we multiply by the conjugate so that we can use a trigonometric identity to convert the cosine in the numerator to a sine:

$$\lim_{\theta\to 0} \frac{1-\cos\theta}{\theta} = \lim_{\theta\to 0} \frac{1-\cos\theta}{\theta} \cdot \frac{1+\cos\theta}{1+\cos\theta}$$

$$= \lim_{\theta\to 0} \frac{1-\cos^2\theta}{\theta(1+\cos\theta)} = \lim_{\theta\to 0} \frac{\sin^2\theta}{\theta(1+\cos\theta)}$$

Now, apply known trigonometric limits:

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} \cdot \lim_{\theta\to 0} \frac{\sin\theta}{1+\cos\theta} = 1 \cdot 0 = 0$$

# Evaluate $\lim_{\theta \to 0} \frac{\sin 3\theta}{\sin 2\theta}$

## Solution

We already know that $\lim_{x \to 0} \frac{\sin x}{x} = 1$. Using $x = 3\theta$ and $x = 2\theta$ and noting that in both cases as $\theta \to 0$, then $x \to 0$, we can conclude that:

$$\lim_{\theta \to 0} \frac{\sin 3\theta}{3\theta} = 1 \quad \text{and} \quad \lim_{\theta \to 0} \frac{\sin 2\theta}{2\theta} = 1$$

Hence, we can determine that:

$$\lim_{\theta \to 0} \frac{\sin 3\theta}{\sin 2\theta} = \lim_{\theta \to 0} \frac{3\theta}{2\theta} \cdot \lim_{\theta \to 0} \frac{\sin 3\theta}{3\theta} \cdot \frac{1}{\lim_{\theta \to 0} \frac{\sin 2\theta}{2\theta}} = \frac{3}{2} \cdot \lim_{\theta \to 0} \frac{\sin 3\theta}{3\theta} \cdot \frac{1}{\lim_{\theta \to 0} \frac{\sin 2\theta}{2\theta}}$$

$$= \frac{3}{2} \cdot 1 \cdot \frac{1}{1} = \frac{3}{2}$$

Therefore:

$$\lim_{\theta \to 0} \frac{1 - \cos \theta}{\theta} = 0$$

# Key Concepts

## Key Concepts

- The limit laws allow us to evaluate limits of functions without having to go through step-by-step processes each time.
- For polynomials and rational functions,

$$\lim_{x \to a} f(x) = f(a)$$

- You can evaluate the limit of a function by factoring and canceling, by multiplying by a conjugate, or by simplifying a complex fraction.
- The Squeeze Theorem allows you to find the limit of a function if the function is always greater than one function and less than another function with limits that are known.

# Key Equations

**Basic Limit Results**

$$\lim_{x \to a} x = a$$

$$\lim_{x \to a} c = c$$

**Important Limits**

$$\lim_{\theta \to 0} \sin\theta = 0$$

$$\lim_{\theta \to 0} \cos\theta = 1$$

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$$

$$\lim_{\theta \to 0} \frac{1 - \cos\theta}{\theta} = 0$$

# Limits at Infinity and Asymptotes

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# Learning Objectives

- Calculate the limit of a function as $x$ increases or decreases without bound.
- Recognize a horizontal asymptote on the graph of a function.
- Estimate the end behavior of a function as $x$ increases or decreases without bound.
- Recognize an oblique asymptote on the graph of a function.

# Definition

## Definition

(Informal) If the values of $f(x)$ become arbitrarily close to $L$ as $x$ becomes sufficiently large, we say the function $f$ has a limit at infinity and write

$$\lim_{x\to\infty} f(x) = L.$$

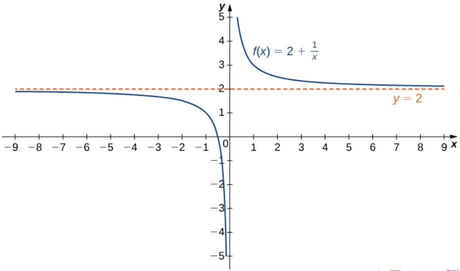
If the values of $f(x)$ become arbitrarily close to $L$ for $x < 0$ as $|x|$ becomes sufficiently large, we say that the function $f$ has a limit at negative infinity and write

$$\lim_{x\to-\infty} f(x) = L.$$

**Figure 1.** The function approaches the asymptote $y = 2$.

| $x$ | 10 | 100 | 1,000 | 10,000 |
|---|---|---|---|---|
| $2 + \frac{1}{x}$ | 2.1 | 2.01 | 2.001 | 2.0001 |
| $x$ | -10 | -100 | -1,000 | -10,000 |
| $2 + \frac{1}{x}$ | 1.9 | 1.99 | 1.999 | 1.9999 |

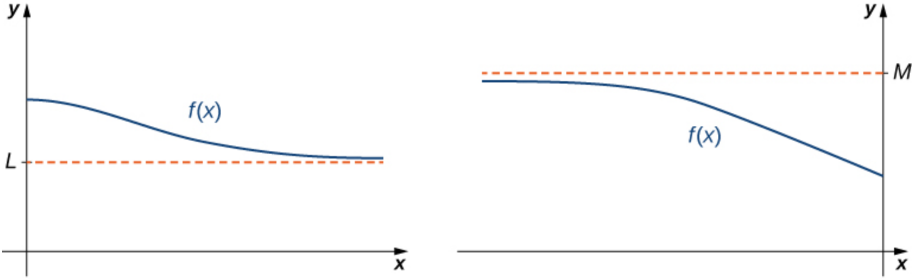
Values of a function $f$ as $x \to \pm\infty$



# Definition

## Horizontal asymptote

If $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$, we say the line $y = L$ is a **horizontal asymptote** of $f$.



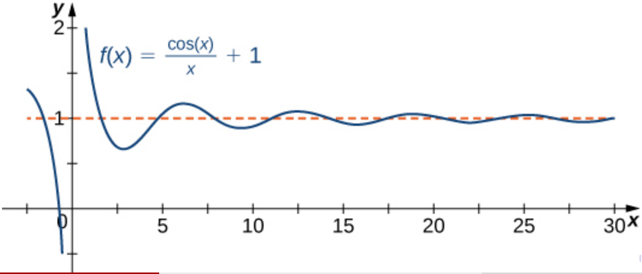

(a) (b)

# Particular case for Horizontal aymptote

A function cannot cross a vertical asymptote because the graph must approach infinity (or $-\infty$) from at least one direction as $x$ approaches the vertical asymptote. However, a function may cross a horizontal asymptote. In fact, a function may cross a horizontal asymptote an unlimited number of times. For example, the function

$$f(x) = \frac{\cos x}{x} + 1$$

intersects the horizontal asymptote $y = 1$ an infinite number of times as it oscillates around the asymptote with ever-decreasing amplitude.



# Example 1

For each of the following functions $f$, we will evaluate $\lim_{x \to \infty} f(x)$ and $\lim_{x \to -\infty} f(x)$ to determine the horizontal asymptote(s).

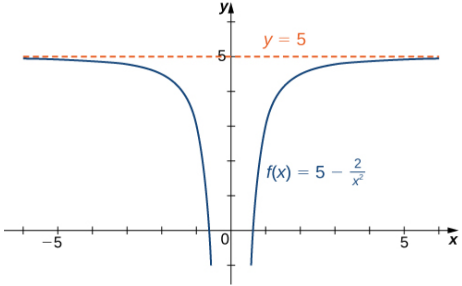
**a.** $f(x) = 5 - \frac{2}{x^2}$

# Example 1

For each of the following functions $f$, we will evaluate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ to determine the horizontal asymptote(s).

**a.** $f(x) = 5 - \frac{2}{x^2}$

- $\lim_{x\to\infty} f(x) = 5 - \frac{2}{\infty} = 5$ and $\lim_{x\to-\infty} f(x) = 5 - \frac{2}{\infty} = 5$
- **Horizontal asymptote:** $y = 5$



# Example 2

**b.** $f(x) = \frac{\sin x}{x}$

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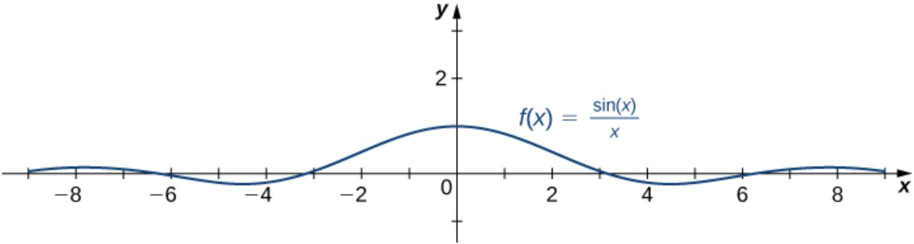
- $\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{\sin x}{x} = 0$ and $\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{\sin x}{x} = 0$
- **Horizontal asymptote:** $y = 0$



# Example 3

**c.** $f(x) = \tan^{-1}(x)$

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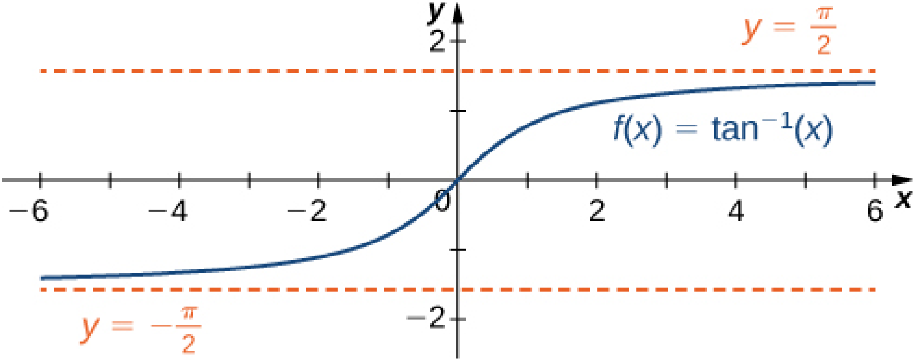
- $\lim_{x\to\infty} f(x) = \frac{\pi}{2}$ and $\lim_{x\to-\infty} f(x) = -\frac{\pi}{2}$
- **Horizontal asymptotes:** $y = \frac{\pi}{2}$ and $y = -\frac{\pi}{2}$



# Computing the Limit of $f(x) = \tan^{-1}(x)$ at Infinity and Negative Infinity

To determine the horizontal asymptotes of the function $f(x) = \tan^{-1}(x)$, we need to evaluate the limits as $x$ approaches $\infty$ and $-\infty$.

1. **Limit as $x \to \infty$:**
   - The function $\tan^{-1}(x)$ (also known as $\arctan(x)$) represents the angle whose tangent is $x$.
   - As $x$ increases towards $\infty$, the angle $\tan^{-1}(x)$ approaches its maximum value, which is $\frac{\pi}{2}$.
   - Therefore,
   $$\lim_{x \to \infty} \tan^{-1}(x) = \frac{\pi}{2}.$$
2. **Limit as $x \to -\infty$:**
   - Similarly, as $x$ decreases towards $-\infty$, the angle $\tan^{-1}(x)$ approaches its minimum value, which is $-\frac{\pi}{2}$.
   - Therefore,
   $$\lim_{x \to -\infty} \tan^{-1}(x) = -\frac{\pi}{2}.$$

Evaluate

$$\lim_{x\to\infty}\left(3+\frac{4}{x}\right) \quad \text{and} \quad \lim_{x\to-\infty}\left(3+\frac{4}{x}\right).$$

Determine the horizontal asymptotes of $f(x) = 3 + \frac{4}{x}$, if any.

# Definition

## Definition

(Informal) We say a function $f$ has an infinite limit at infinity and write

$$\lim_{x \to \infty} f(x) = \infty.$$

if $f(x)$ becomes arbitrarily large for $x$ sufficiently large. We say a function has a negative infinite limit at infinity and write

$$\lim_{x \to \infty} f(x) = -\infty.$$

if $f(x) < 0$ and $|f(x)|$ becomes arbitrarily large for $x$ sufficiently large. Similarly, we can define infinite limits as $x \to -\infty$.

# Definition

## Formal Definition

We say a function $f$ has a limit at infinity if there exists a real number $L$ such that for all $\epsilon > 0$, there exists $N > 0$ such that

$$|f(x) - L| < \epsilon \quad \text{for all } x > N.$$

In that case, we write

$$\lim_{x \to \infty} f(x) = L.$$

We say a function $f$ has a limit at negative infinity if there exists a real number $L$ such that for all $\epsilon > 0$, there exists $N < 0$ such that

$$|f(x) - L| < \epsilon \quad \text{for all } x < N.$$

In that case, we write

$$\lim_{x \to -\infty} f(x) = L.$$

# Graph of Limit

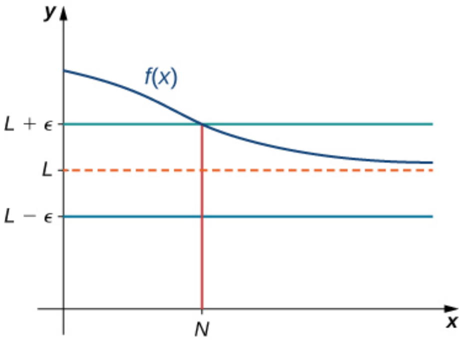



Figure: $|f(x) - L| < \epsilon$ for all $x < N$.

# A Finite Limit at Infinity Example

Use the formal definition of limit at infinity to prove that

$$\lim_{x\to\infty}\left(2+\frac{1}{x}\right)=2.$$

## Solution

Let $\epsilon > 0$. Let $N = \frac{1}{\epsilon}$. Therefore, for all $x > N$, we have

$$\left|2+\frac{1}{x}-2\right| = \left|\frac{1}{x}\right| = \frac{1}{x} < \frac{1}{N} = \epsilon.$$

# A Finite Limit at Infinity Example

Use the formal definition of limit at infinity to prove that

$$\lim_{x \to \infty} \left(3 - \frac{1}{x^2}\right) = 3.$$

## Hint

Let $N = \frac{1}{\sqrt{\epsilon}}$.

# A Finite Limit at Infinity Example

Use the formal definition of limit at infinity to prove that

$$\lim_{x\to\infty}\left(3-\frac{1}{x^2}\right)=3.$$

## Hint

Let $N=\frac{1}{\sqrt{\epsilon}}$.

## Solution

Let $\epsilon>0$. Let $N=\frac{1}{\sqrt{\epsilon}}$. Therefore, for all $x>N$, we have

$$\left|3-\frac{1}{x^2}-3\right|=\left|\frac{1}{x^2}\right|<\frac{1}{N^2}=\epsilon.$$

Therefore,

$$\lim_{x\to\infty}\left(3-\frac{1}{x^2}\right)=3.$$

# Definition

## Formal Definition

We say a function $f$ has an infinite limit at infinity and write

$$\lim_{x \to \infty} f(x) = \infty$$

if for all $M > 0$, there exists an $N > 0$ such that

$$f(x) > M \quad \text{for all } x > N.$$

We say a function has a negative infinite limit at infinity and write

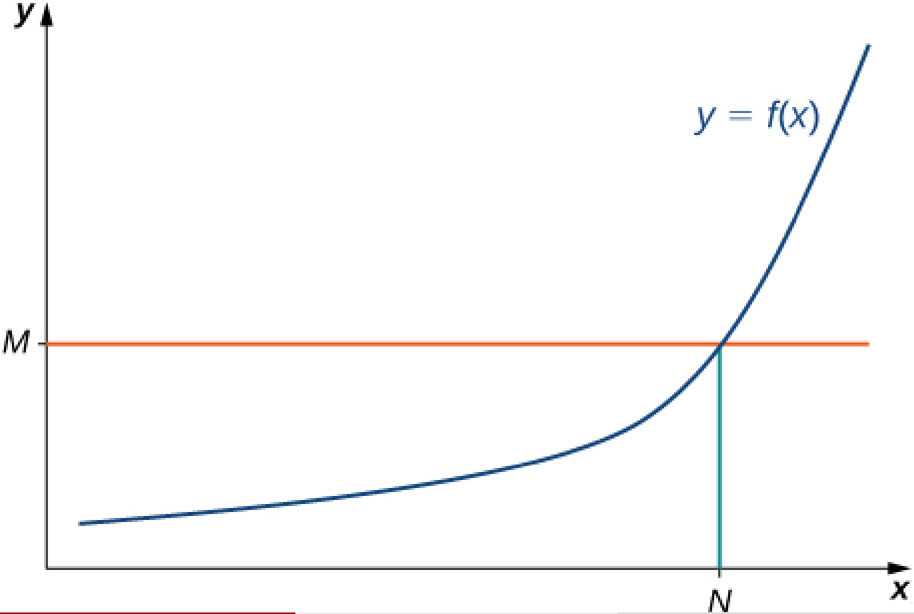
$$\lim_{x \to \infty} f(x) = -\infty$$

if for all $M < 0$, there exists an $N > 0$ such that

$$f(x) < M \quad \text{for all } x > N.$$

Similarly, we can define limits as $x \to -\infty$.

# Infinity limit graph



# An Infinite Limit at Infinity

Use the formal definition of infinite limit at infinity to prove that

$$\lim_{x\to\infty} x^3 = \infty.$$

## Solution

Let $M > 0$. Let $N = \sqrt[3]{M}$. Then, for all $x > N$, we have

$$x^3 > M.$$

Therefore, $\lim_{x\to\infty} x^3 = \infty$.

# An Infinite Limit at Infinity

Use the formal definition of infinite limit at infinity to prove that

$$\lim_{x \to \infty} 3x^2 = \infty.$$

## Hint

Let $N = \sqrt{\frac{M}{3}}$.

## Solution

Let $M > 0$. Let $N = \sqrt{\frac{M}{3}}$. Then, for all $x > N$, we have

$$3x^2 > M.$$

Therefore, $\lim_{x \to \infty} 3x^2 = \infty$.

# Key Concepts

- The limit of $f(x)$ is $L$ as $x \to \infty$ (or as $x \to -\infty$) if the values $f(x)$ become arbitrarily close to $L$ as $x$ becomes sufficiently large.
- The limit of $f(x)$ is $\infty$ as $x \to \infty$ if $f(x)$ becomes arbitrarily large as $x$ becomes sufficiently large. The limit of $f(x)$ is $-\infty$ as $x \to \infty$ if $f(x) < 0$ and $|f(x)|$ becomes arbitrarily large as $x$ becomes sufficiently large. We can define the limit of $f(x)$ as $x$ approaches $-\infty$ similarly.

# Continuity

Clotilde Djuikem

# Learning Objectives

- Explain the three conditions for continuity at a point.
- Describe three kinds of discontinuities.
- Define continuity on an interval.
- State the theorem for limits of composite functions.
- Provide an example of the Intermediate Value Theorem.

# Continuity at a Point

**Definition**

A function $f(x)$ is continuous at a point $a$ if and only if the following three conditions are satisfied:

1. $f(a)$ is defined.
2. $\lim_{x \to a} f(x)$ exists.
3. $\lim_{x \to a} f(x) = f(a)$.

# Problem-Solving Strategy: Determining Continuity at a Point

**1. Check to see if $f(a)$ is defined.** If $f(a)$ is undefined, we need go no further. The function is not continuous at $a$. If $f(a)$ is defined, continue to step 2.

**2. Compute $\lim_{x \to a} f(x)$.** In some cases, we may need to do this by first computing $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$. If $\lim_{x \to a} f(x)$ does not exist (that is, it is not a real number), then the function is not continuous at $a$ and the problem is solved. If $\lim_{x \to a} f(x)$ exists, then continue to step 3.

**3. Compare $f(a)$ and $\lim_{x \to a} f(x)$.** If $\lim_{x \to a} f(x) \neq f(a)$, then the function is not continuous at $a$. If $\lim_{x \to a} f(x) = f(a)$, then the function is continuous at $a$.

# Example: Continuity at a Point

**Problem:** Using the definition, determine whether the function

$$f(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

is continuous at $x = 0$.

**Solution:**

- First, observe that $f(0) = 1$.
- Next,
$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\sin x}{x} = 1.$$
- Last, compare $f(0)$ and $\lim_{x\to 0} f(x)$. We see that
$$f(0) = 1 = \lim_{x\to 0} f(x).$$
- Since all three of the conditions in the definition of continuity are satisfied, $f(x)$ is continuous at $x = 0$.

# Example: Continuity at a Point

**Problem:** Using the definition, determine whether the function

$$f(x) = \begin{cases} 2x+1 & \text{if } x < 1 \\ 2 & \text{if } x = 1 \\ -x+4 & \text{if } x > 1 \end{cases}$$

is continuous at $x = 1$. If the function is not continuous at 1, indicate the condition for continuity at a point that fails to hold.

**Solution:**

- First, calculate $f(1)$: $f(1) = 2$.
- Next, compute $\lim_{x \to 1^-} f(x)$ and $\lim_{x \to 1^+} f(x)$:

  $$\lim_{x \to 1^-} (2x+1) = 3. \textbf{ and } \lim_{x \to 1^+} (-x+4) = 3.$$

  Since $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = 3$, we have: $\lim_{x \to 1} f(x) = 3$.
- Compare $f(1)$ with $\lim_{x \to 1} f(x)$: $f(1) = 2 \quad \text{and} \quad \lim_{x \to 1} f(x) = 3$.
- Since $f(1) \neq \lim_{x \to 1} f(x)$, the function is not continuous at $x = 1$.

# Continuity of Polynomials and Rational Functions

**Theorem**

Polynomials and rational functions are continuous at every point in their domains.

**Example:** Determine the points of discontinuity for $f(x) = \frac{x+1}{x-5}$.

- $f(x)$ is continuous for all $x \neq 5$.

# Continuity on an Interval

**Definition**

A function $f(x)$ is continuous over an interval if it is continuous at every point in that interval. For a closed interval $[a, b]$, $f(x)$ must also be continuous from the right at $a$ and from the left at $b$.

**Example:** Determine the intervals over which $f(x) = \sqrt{4 - x^2}$ is continuous.

- $f(x)$ is continuous over the interval $[-2, 2]$.

# Example: Continuity at a Point

**Problem:** Using the definition, determine whether the function

$$f(x) = \begin{cases} -x^2 + 4 & \text{if } x \leq 3 \\ 4x - 8 & \text{if } x > 3 \end{cases}$$

is continuous at $x = 3$. Justify the conclusion.

**Solution:**

- Let's begin by trying to calculate $f(3)$:

  $$f(3) = -(3)^2 + 4 = -5.$$

  Thus, $f(3)$ is defined. Next, we calculate $\lim_{x \to 3} f(x)$. To do this, we must compute $\lim_{x \to 3^-} f(x)$ and $\lim_{x \to 3^+} f(x)$:
- $$\lim_{x \to 3^-} f(x) = -(3)^2 + 4 = -5 \textbf{ and } \lim_{x \to 3^+} f(x) = 4(3) - 8 = 4.$$
- Therefore, $\lim_{x \to 3} f(x)$ does not exist. Thus, $f(x)$ is not continuous at 3.

# Types of Discontinuities

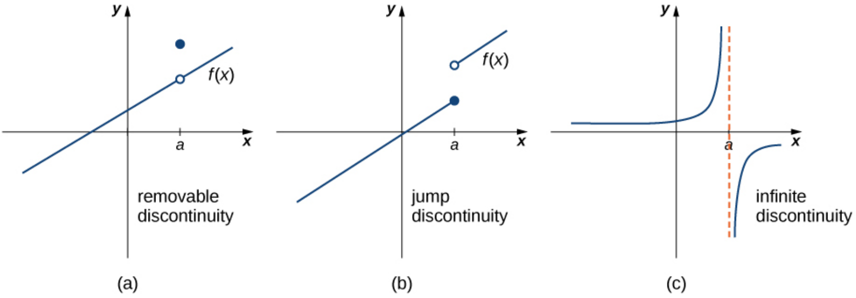
**Definition**

- **Removable Discontinuity:** A discontinuity at $a$ where $\lim_{x\to a} f(x)$ exists but $f(a)$ is not defined or $f(a) \neq \lim_{x\to a} f(x)$.
- **Jump Discontinuity:** A discontinuity at $a$ where $\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$.
- **Infinite Discontinuity:** A discontinuity at $a$ where $\lim_{x\to a} f(x)$ is $\infty$ or $-\infty$.

**Example:** For $f(x) = \frac{x+2}{x+1}$, identify the discontinuity at $x = -1$.

- The function $f(x)$ has an infinite discontinuity at $x = -1$ because $\lim_{x\to -1} f(x) = \pm\infty$.

# Types of Discontinuities



**Problem:**

$$f(x) = \frac{x^2 - 4}{x - 2}$$

Classify this discontinuity as removable, jump, or infinite.

**Solution:**
To classify the discontinuity at 2, we must evaluate $\lim_{x \to 2} f(x)$:

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{(x-2)(x+2)}{x - 2} = \lim_{x \to 2} (x + 2) = 4$$

Since $f$ is discontinuous at 2 and $\lim_{x \to 2} f(x)$ exists, $f$ has a removable discontinuity at $x = 2$.

# Classifying a Discontinuity

**Problem:** In (Figure), we showed that

$$f(x) = \begin{cases} -x^2 + 4 & \text{if } x \leq 3 \\ 4x - 8 & \text{if } x > 3 \end{cases}$$

is discontinuous at $x = 3$. Classify this discontinuity as removable, jump, or infinite.

**Solution:**
Earlier, we showed that $f$ is discontinuous at 3 because $\lim_{x \to 3} f(x)$ does not exist. However, since

$$\lim_{x \to 3^-} f(x) = -5 \textbf{ and } \lim_{x \to 3^+} f(x) = 4$$

both exist, we conclude that the function has a jump discontinuity at 3.

# Composite Function Theorem

**Theorem**

If $f(x)$ is continuous at $L$ and $\lim_{x\to a} g(x) = L$, then

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right) = f(L).$$

**Example:** Evaluate

$$\lim_{x\to\pi/2} \cos\left(x - \frac{\pi}{2}\right).$$

**Solution:**

The given function is a composite of $\cos x$ and $x - \frac{\pi}{2}$. Since

$$\lim_{x\to\pi/2} \left(x - \frac{\pi}{2}\right) = 0$$

and $\cos x$ is continuous at 0, we may apply the composite function theorem. Thus,

$$\lim_{x\to\pi/2} \cos\left(x - \frac{\pi}{2}\right) = \cos\left(\lim_{x\to\pi/2} \left(x - \frac{\pi}{2}\right)\right) = \cos(0) = 1.$$

# Limit of a Sine Function

**Problem:** Evaluate

$$\lim_{x \to \pi} \sin(x - \pi).$$

**Solution:**

- The given function is a composite of the sine function and $x - \pi$.
- First, calculate the inner limit:

$$\lim_{x \to \pi} (x - \pi) = 0.$$

- Since the sine function $\sin x$ is continuous for all real numbers, we can use the composite function theorem. Thus, we can substitute the limit of the inner function into the sine function:

$$\lim_{x \to \pi} \sin(x - \pi) = \sin\left(\lim_{x \to \pi} (x - \pi)\right) = \sin(0).$$

- Now, evaluate $\sin(0)$:

$$\sin(0) = 0.$$

- Therefore,

$$\lim_{x \to \pi} \sin(x - \pi) = 0.$$

# Continuity of Trigonometric Functions

## Continuity

Trigonometric functions are continuous over their entire domains.

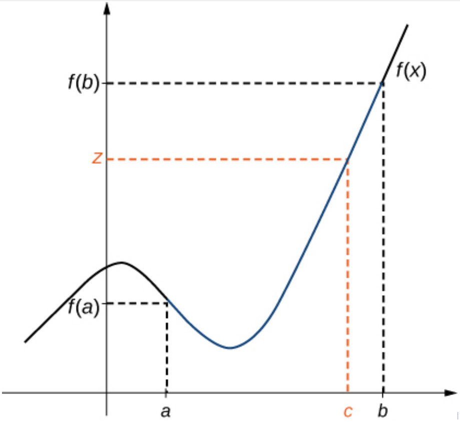
## Continuityon an Interval

If a polynomial, rational, trigonometric, inverse trigonometric, exponential, logarithmic, or radical function is defined on an interval, then it is continuous on that interval.

# Intermediate Value Theorem

**Theorem**

If $f$ is continuous on a closed interval $[a, b]$ and $z$ is any real number between $f(a)$ and $f(b)$, then there exists a number $c \in [a, b]$ such that $f(c) = z$.



# Application of the Intermediate Value Theorem

**Problem:** Show that

$$f(x) = x - \cos x$$

has at least one zero.

**Solution:**

- Since $f(x) = x - \cos x$ is continuous over $(-\infty, +\infty)$, it is continuous over any closed interval of the form $[a, b]$. If you can find an interval $[a, b]$ such that $f(a)$ and $f(b)$ have opposite signs, you can use the Intermediate Value Theorem to conclude there must be a real number $c$ in $(a, b)$ that satisfies $f(c) = 0$.
- Note that

$$f(0) = 0 - \cos(0) = -1 < 0$$

and

$$f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} - \cos\left(\frac{\pi}{2}\right) = \frac{\pi}{2} > 0.$$

- Using the Intermediate Value Theorem, we can see that there must be a real number $c$ in $[0, \pi/2]$ that satisfies $f(c) = 0$. Therefore, $f(x) = x - \cos x$ has at least one zero.

# When Can You Apply the Intermediate Value Theorem?

**Problem:** If $f(x)$ is continuous over $[0, 2]$, $f(0) > 0$, and $f(2) > 0$, can we use the Intermediate Value Theorem to conclude that $f(x)$ has no zeros in the interval $[0, 2]$? Explain.

**Solution:**

- No. The Intermediate Value Theorem only allows us to conclude that we can find a value between $f(0)$ and $f(2)$; it doesn't allow us to conclude that we can't find other values.
- To see this more clearly, consider the function

$$f(x) = (x - 1)^2.$$

  It satisfies

$$f(0) = 1 > 0, \quad f(2) = 1 > 0,$$

  and

$$f(1) = 0.$$

- This function has a zero at $x = 1$ despite $f(0) > 0$ and $f(2) > 0$. Thus, we cannot conclude that $f(x)$ has no zeros in the interval $[0, 2]$.

# Key Concepts

- A function is continuous at a point if it is defined, its limit exists, and the limit equals the function value.
- Discontinuities can be classified as removable, jump, or infinite.
- The Composite Function Theorem and Intermediate Value Theorem help establish the continuity of more complex functions.
- Continuity is essential for analyzing the behavior of functions over intervals.