7.1 Parametric Equations

Math 1700

University of Manitoba

Winter 2024

Outline

Learning Objectives

- Plot a curve described by parametric equations.
- Convert the parametric equations of a curve into the form $y = f(x)$.
- Recognize the parametric equations of basic curves, such as a line and a circle.
- Recognize the parametric equations of a cycloid.

Definition

If x and y are continuous functions of t on an interval I, then the equations

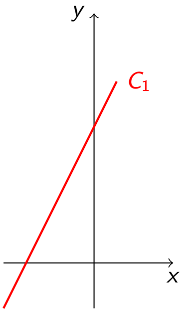
$$x = x(t) \quad \text{and} \quad y = y(t)$$

are called parametric equations and t is called the parameter. The set of points (x, y) obtained as t varies over the interval I is called the graph of the parametric equations. The graph of parametric equations is also called a parametric curve or plane curve, and is denoted by C.

Graphing Parametrically Defined Curves

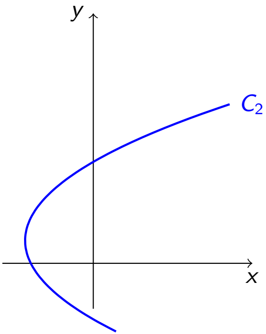
1st:

$$x(t) = t - 1$$
$$y(t) = 2t + 4$$
$$-3 \leq t \leq 2$$



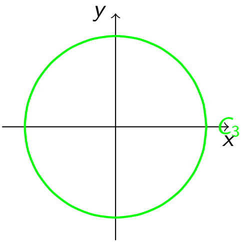
2nd:

$$x(t) = t^2 - 3$$
$$y(t) = 2t + 1$$
$$-2 \leq t \leq 3$$



3rd:

$$x(t) = 4\cos t$$
$$y(t) = 4\sin t$$
$$0 \leq t \leq 2\pi$$



Example 1

Sketch the curves described by the following parametric equations:

$$x(t) = t - 1, \quad y(t) = 2t + 4, \quad -3 \le t \le 2$$

To create a graph of this curve, first set up a table of values. Since the independent variable in both $x(t)$ and $y(t)$ is t, let t appear in the first column.

t	$x(t)$	$y(t)$
-3	-4	-2
-2	-3	0
-1	-2	2
0	-1	4
1	0	6
2	1	8

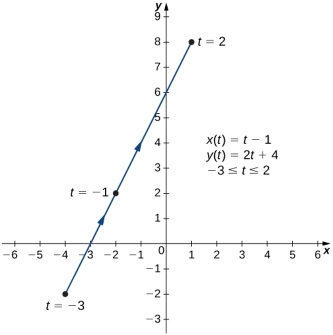
Example 1

Sketch the curves described by the following parametric equations:

$$x(t) = t - 1, \quad y(t) = 2t + 4, \quad -3 \le t \le 2$$

To create a graph of this curve, first set up a table of values. Since the independent variable in both $x(t)$ and $y(t)$ is t, let t appear in the first column.

t	$x(t)$	$y(t)$
-3	-4	-2
-2	-3	0
-1	-2	2
0	-1	4
1	0	6
2	1	8



The arrows on the graph indicate the orientation of the graph, that is, the direction that a point moves on the graph as t varies from -3 to 2.

Example 2

Sketch the curves described by the following parametric equations:

$$x(t) = t^2 - 3,\ y(t) = 2t + 1,\ -2 \leq t \leq 3$$

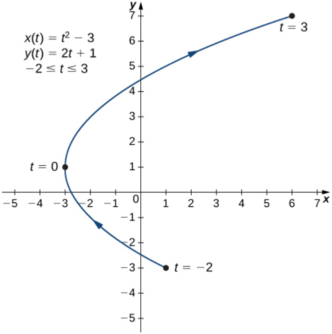
t	$x(t)$	$y(t)$
-2	1	-3
-1	-2	-1
0	-3	1
1	-2	3
2	1	5
3	6	7

Example 2

Sketch the curves described by the following parametric equations:

$$x(t) = t^2 - 3,\ y(t) = 2t + 1,\ -2 \leq t \leq 3$$

t	$x(t)$	$y(t)$
-2	1	-3
-1	-2	-1
0	-3	1
1	-2	3
2	1	5
3	6	7



As t progresses from -2 to 3, the point on the curve travels along a parabola. The direction the point moves is again called the orientation and is indicated on the graph.

Example 3

Sketch the curves described by the following parametric equations:

$$x(t) = 4\cos t,\ y(t) = 4\sin t,\ 0 \le t \le 2\pi$$

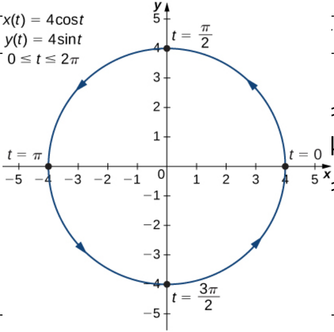
t	$x(t)$	$y(t)$	t	$x(t)$	$y(t)$
0	4	0	$\frac{7\pi}{6}$	$-2\sqrt{3} \approx -3.5$	2
$\frac{\pi}{6}$	$2\sqrt{3} \approx 3.5$	2	2	$\frac{4\pi}{3}$	$-2\sqrt{3} \approx -3.5$
$\frac{\pi}{3}$	2	$2\sqrt{3} \approx 3.5$	$\frac{3\pi}{2}$	0	-4
$\frac{\pi}{2}$	0	4	$\frac{5\pi}{3}$	2	$-2\sqrt{3} \approx -3.5$
$\frac{2\pi}{3}$	-2	$2\sqrt{3} \approx 3.5$	$\frac{11\pi}{6}$	$2\sqrt{3} \approx 3.5$	2
$\frac{5\pi}{6}$	$-2\sqrt{3} \approx -3.5$	2	2π	4	0
π	-4	0			

Example 3

Sketch the curves described by the following parametric equations:

$$x(t) = 4\cos t,\ y(t) = 4\sin t,\ 0 \le t \le 2\pi$$

t	$x(t)$	$y(t)$	t	)
0	4	0	$\frac{7\pi}{6}$	
$\frac{\pi}{6}$	$2\sqrt{3} \approx 3.5$	2	2	-3.5
$\frac{\pi}{3}$	2	$2\sqrt{3} \approx 3.5$	$\frac{3\pi}{2}$	
$\frac{\pi}{2}$	0	4	$\frac{5\pi}{3}$	-3.5
$\frac{2\pi}{3}$	-2	$2\sqrt{3} \approx 3.5$	$\frac{11\pi}{6}$	
$\frac{5\pi}{6}$	$-2\sqrt{3} \approx -3.5$	2	2π	
π	-4	0		

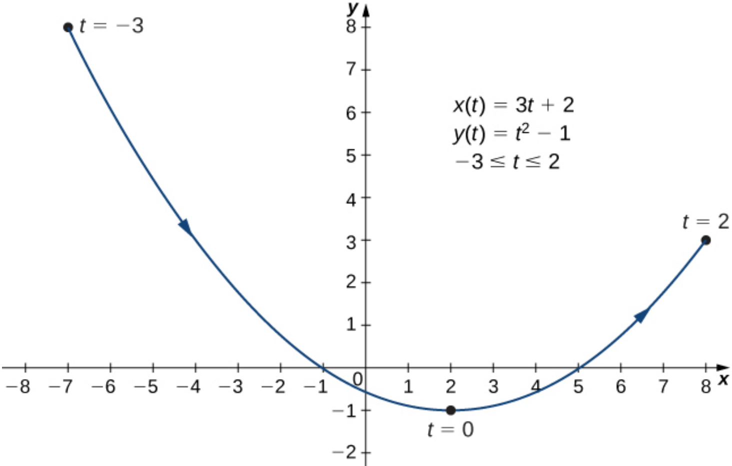


This is the graph of a circle with radius 4 centered at the origin, with a counterclockwise orientation. The starting point and ending point of the curve both have coordinates $(4, 0)$.

Sketching the Curve

Sketch the curve described by the parametric equations:

$$x(t) = 3t + 2, \quad y(t) = t^2 - 1, \quad -3 \le t \le 2$$



Rewriting Parametric Equations

To better understand the graph of a curve represented parametrically, it is useful to rewrite the two equations as a single equation relating the variables x and y. For example, consider the parametric equations:

$$x(t) = t^2 - 3, \quad y(t) = 2t + 1, \quad -2 \leq t \leq 3$$

Solving the second equation for t gives:

$$t = \frac{y-1}{2}$$

This can be substituted into the first equation:

$$x = \left(\frac{y-1}{2}\right)^2 - 3 = \frac{(y^2 - 2y + 1)}{4} - 3 = \frac{y^2 - 2y - 11}{4}$$

This equation describes x as a function of y.

Eliminating the Parameter: Example 1

Eliminate the parameter for each of the plane curves described by the following parametric equations and describe the resulting graph:

$$x(t) = \sqrt{2t+4}, \quad y(t) = 2t+1, \quad -2 \leq t \leq 6$$

Solution: To eliminate the parameter, we can solve either of the equations for t. For example, solving the first equation for t gives:

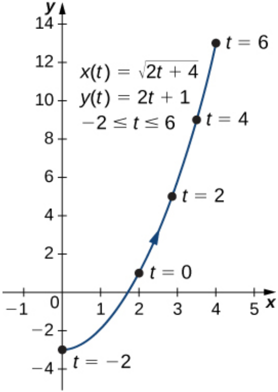
$$x = \sqrt{2t+4} \Rightarrow t = \frac{x^2-4}{2}$$

Note that when we square both sides, it is important to observe that $x \geq 0$. Substituting $t = \frac{x^2-4}{2}$ into $y(t)$ yields:

$$y(t) = 2t+1 = 2\left(\frac{x^2-4}{2}\right)+1 = x^2-4+1 = x^2-3$$

Part 2 example 1

$y = x^2 - 3$, this is the equation of a parabola opening upward. There is, however, a domain restriction because of the limits on the parameter t. When $t = -2$, $x = \sqrt{2(-2)+4} = 0$, and when $t = 6$, $x = \sqrt{2(6)+4} = 4$. The graph of this plane curve follows.



Eliminating the Parameter Creatively

Sometimes it is necessary to be a bit creative in eliminating the parameter.

$$x(t) = 4\cos(t) \quad \text{and} \quad y(t) = 3\sin(t).$$

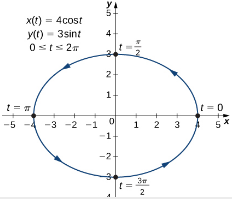
Solving either equation for t directly is not advisable because sine and cosine are not one-to-one functions.

$$\cos(t) = \frac{x}{4} \quad \text{and} \quad \sin(t) = \frac{y}{3}.$$

Using the Pythagorean identity $\cos^2(t) + \sin^2(t) = 1$, we obtain:

$$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{3}\right)^2 = 1.$$

This is the equation of a horizontal ellipse centered at the origin, with semimajor axis 4 and semiminor axis 3.



Eliminating the Parameter

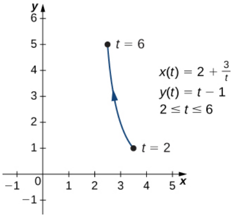
To eliminate the parameter for the plane curve defined by the parametric equations:

$$x(t) = 2 + \frac{3}{t}, \quad y(t) = t - 1, \quad 2 \leq t \leq 6$$

We can express t in terms of x and y: $x = 2 + \frac{3}{t} \implies t = \frac{3}{x-2}$. Then substitute t into the equation for y:

$$y = \frac{3}{x-2} - 1 \implies y = -1 + \frac{3}{x-2}$$

So, we have $x = 2 + \frac{3}{y+1}$ or $y = -1 + \frac{3}{x-2}$. This equation describes a portion of a rectangular hyperbola centered at $(2, -1)$.



Parameterizing a Curve

Find two different pairs of parametric equations to represent the graph of $y = 2x^2 - 3$:

1. **First Parametric Equations:**

$$x(t) = t, \quad y(t) = 2t^2 - 3$$

   Since there is no restriction on the domain in the original graph, there is no restriction on the values of t.

2. **Second Parametric Equations:**

$$x(t) = 3t - 2, \quad y(t) = 18t^2 - 24t + 6$$

   We have complete freedom in the choice for the second parameterization. We can choose $x(t) = 3t - 2$ since there are no restrictions imposed on x, and then substitute it into the equation $y = 2x^2 - 3$.

Therefore, the second parameterization of the curve can be written as:

$$x(t) = 3t - 2, \quad y(t) = 18t^2 - 24t + 6$$

Parameterizing a Curve

Find two different sets of parametric equations to represent the graph of $y = x^2 + 2x$:

1. **First Parametric Equations:**

$$x(t) = t, \quad y(t) = t^2 + 2t$$

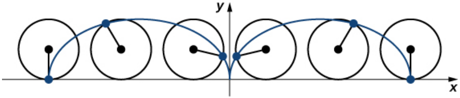
2. **Second Parametric Equations:**

$$x(t) = 2t - 3, \quad y(t) = (2t-3)^2 + 2(2t-3) = 4t^2 - 8t + 3$$

There are, in fact, an infinite number of possibilities.

The Cycloid: Nature's Artistry

- Imagine embarking on a tranquil bicycle ride through the countryside, where every rotation of the tire leaves a rhythmic mark on the road.
- Picture a determined ant seeking its way home after a long day's journey, hitchhiking along the tire's edge for a free ride.
- The path traced by this intrepid ant on a straight road is what we call a cycloid.



The Cycloid: Parameterizing

Parametric equations

A cycloid generated by a circle (or bicycle wheel) of radius a is given by the parametric equations

$$x(t) = a(t - \sin t), \quad y(t) = a(1 - \cos t)$$

Proof

If the radius is a, then the coordinates of the center can be given by the equations

$$x(t) = at, \quad y(t) = a$$

A possible parameterization of the circular motion of the ant (relative to the center of the wheel) is given by

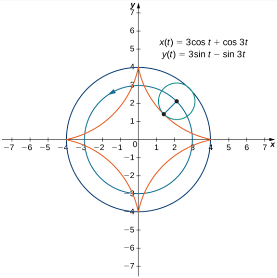
$$x(t) = -a \sin t, \quad y(t) = -a \cos t$$

Adding these equations together gives the equations for the cycloid.

The hypocycloid.: Large wheel

Visualizing

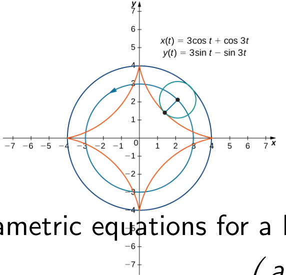
Suppose that the bicycle wheel doesn't travel along a straight road but instead moves along the inside of a larger wheel, as in Figure. A point on the edge of the green circle traces out the red graph, which is called a hypocycloid.



The hypocycloid.: Large wheel

Visualizing

Suppose that the bicycle wheel doesn't travel along a straight road but instead moves along the inside of a larger wheel, as in Figure. A point on the edge of the green circle traces out the red graph, which is called a hypocycloid.



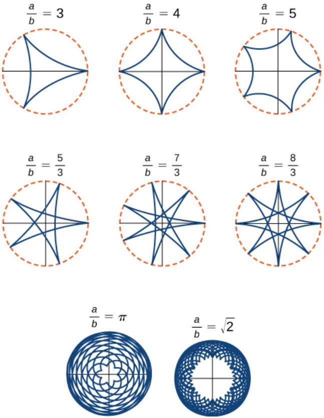
The general parametric equations for a hypocycloid are:

$$x(t) = (a-b)\cos t + b\cos\left(\frac{a-b}{b}t\right)$$

$$y(t) = (a-b)\sin t - b\sin\left(\frac{a-b}{b}t\right)$$

Examples Hypocycloid

The period of the second trigonometric function in both $x(t)$ and $y(t)$ is equal to $\frac{2\pi b}{a-b}$. The ratio $\frac{a}{b}$is related to the number of cusps (corners or pointed ends) on the graph, as illustrated in the following Figure



Key Concepts: Parametric Equations

- Parametric equations provide a convenient way to describe a curve. A parameter can represent time or some other meaningful quantity.
- It is often possible to eliminate the parameter in a parameterized curve to obtain a function or relation describing that curve.
- There is always more than one way to parameterize a curve.
- Parametric equations can describe complicated curves that are difficult or perhaps impossible to describe using rectangular coordinates.

7.2 Calculus of Parametric Curves

Math 1700

University of Manitoba

Winter 2024

Outline

Learning Objectives

1. Determine derivatives and equations of tangents for parametric curves.
2. Find the area under a parametric curve.
3. Determine the arc length of a parametric curve.
4. Apply the formula for the surface area of the surface generated by revolving a parametric curve about the x-axis or the y-axis.

Challenge

Challenge

Now that we have introduced the concept of a parameterized curve, our next step is to learn how to work with this concept in the context of calculus. For example, if we know a parameterization of a given curve, is it possible to calculate the slope of a tangent line to the curve? How about the arc length of the curve? Or the area under the curve?

Derivative of Parametric Equations

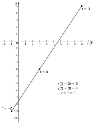
Theorem

Consider the plane curve defined by the parametric equations $x = x(t)$ and $y = y(t)$. Suppose that $x'(t)$ and $y'(t)$ exist, and assume that $x'(t) \neq 0$. Then the derivative $\frac{dy}{dx}$ is given by

$$(*) \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{y'(t)}{x'(t)}.$$

Derivative of Parametric Equations: Example

Let



$$x(t) = 2t + 3,$$
$$y(t) = 3t - 4, \quad -2 \le t \le 3.$$

It is a line segment starting at $(-1, -10)$ and ending at $(9, 5)$. We can eliminate the parameter by first solving the equation $x(t) = 2t + 3$ for t:

$$x(t) = 2t + 3,\ x - 3 = 2t,\ t = \frac{x-3}{2}.$$

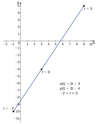
Substituting this into $y(t)$, we obtain:

$$y(t) = 3t - 4,\ y = 3\left(\frac{x-3}{2}\right) - 4,\ y = \frac{3x}{2} - \frac{9}{2} - 4,\ y = \frac{3x}{2} - \frac{17}{2}.$$

The slope of this line is given by $\frac{dy}{dx} = \frac{3}{2}$.

Derivative of Parametric Equations: Example

Let



$$x(t) = 2t + 3,$$
$$y(t) = 3t - 4, \quad -2 \leq t \leq 3.$$

It is a line segment starting at $(-1, -10)$ and ending at $(9, 5)$. We can eliminate the parameter by first solving the equation $x(t) = 2t + 3$ for t:

$$x(t) = 2t + 3,\ x - 3 = 2t,\ t = \frac{x-3}{2}.$$

Substituting this into $y(t)$, we obtain:

$$y(t) = 3t - 4,\ y = 3\left(\frac{x-3}{2}\right) - 4,\ y = \frac{3x}{2} - \frac{9}{2} - 4,\ y = \frac{3x}{2} - \frac{17}{2}.$$

The slope of this line is given by $\frac{dy}{dx} = \frac{3}{2}$.

Using Theorem, we calculate $x'(t)$ and $y'(t)$, which gives $x'(t) = 2$ and $y'(t) = 3$. Notice that $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3}{2}$.

Example 1

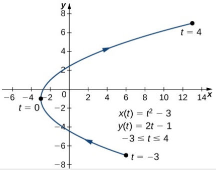
For the parametric equations:

$$x(t) = t^2 - 3, \; y(t) = 2t - 1, \quad -3 \leq t \leq 4,$$

we first calculate $\frac{dx}{dt}$ and $\frac{dy}{dt}$: $\frac{dx}{dt} = 2t, \quad \frac{dy}{dt} = 2.$
Substituting these into $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2}{2t} = \frac{1}{t}.$$

Since $\frac{dy}{dt} \neq 0$, there are no points where the tangent line is horizontal. Solving $\frac{dx}{dt} = 2t = 0$, we find $t = 0$, corresponding to the point $(-3, -1)$ on the curve.



Example 2

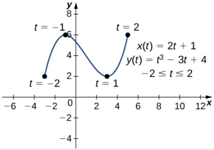
For the parametric equations:

$$x(t) = 2t + 1,\ y(t) = t^3 - 3t + 4, \quad -2 \le t \le 2,$$

we first calculate $\frac{dx}{dt}$ and $\frac{dy}{dt}$: $\frac{dx}{dt} = 2, \quad \frac{dy}{dt} = 3t^2 - 3.$
Substituting these into $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t^2 - 3}{2}.$$

Since $\frac{dx}{dt} \neq 0$, there are no points where the tangent line is vertical. To find where the tangent line is horizontal, we solve $\frac{dy}{dt} = 3t^2 - 3 = 0$, giving $t = \pm 1$. At $t = -1$, the point $(-1, 6)$ is on the curve, and at $t = 1$, the point $(3, 2)$ is on the curve.



Example 3

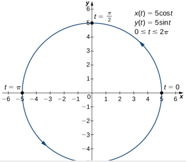
For the parametric equations:

$$x(t) = 5\cos(t),\ y(t) = 5\sin(t), \quad 0 \le t \le 2\pi,$$

we first calculate $\frac{dx}{dt}$ and $\frac{dy}{dt}$: $\frac{dx}{dt} = -5\sin(t), \quad \frac{dy}{dt} = 5\cos(t)$.
Substituting these into $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = -\cot(t).$$

Points where $\frac{dy}{dt} = 0$ occur at $t = \frac{\pi}{2}$ and $t = \frac{3\pi}{2}$ in the interval $[0, 2\pi]$.
Solving $\frac{dx}{dt} = -5\sin(t) = 0$ yields $t = 0, \pi, 2\pi$. The points corresponding to these values are $(5,0), (-5,0), (5,0)$ respectively.



Derivative and Tangent Lines

Calculate the derivative $\frac{dy}{dx}$ for the curve defined by the parametric equations

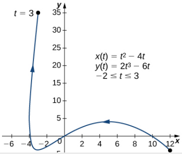
$$x(t) = t^2 - 4t, \quad y(t) = 2t^3 - 6t, \quad -2 \leq t \leq 3$$

and find all points on the curve where the tangent line is horizontal or vertical.

Answer:

$$\frac{dy}{dx} = \frac{6t^2 - 6}{2t - 4} = \frac{3t^2 - 3}{t - 2}$$

The tangent line is horizontal at $(-3, 4)$ and $(5, 4)$, corresponding to $t = 1$ and $t = -1$ respectively. The tangent line is vertical at $(-4, 4)$, corresponding to $t = 2$.



Slope of the Tangent Line in a Special Case

Determine the slope of the tangent line to the hypocycloid

$$x(t) = 3\cos(t) + \cos(3t), \quad y(t) = 3\sin(t) - \sin(3t)$$

at the point corresponding to $t = 0$.
Solution: We first calculate $x'(t)$ and $y'(t)$:

$$x'(t) = -3\sin(t) - 3\sin(3t), \quad y'(t) = 3\cos(t) - 3\cos(3t).$$

We see that $x'(0) = 0$, and so $(*)$ cannot be applied to find $\frac{dy}{dx}$ when $t = 0$. However, $x'(t) \neq 0$ when $t \in [-\frac{\pi}{6}, \frac{\pi}{6}] \setminus \{0\}$, $x'(t) > 0$ when $t \in \left[-\frac{\pi}{6}, 0\right)$ and $x'(t) < 0$ when $t \in (0, \frac{\pi}{6}])$, and so we can consider

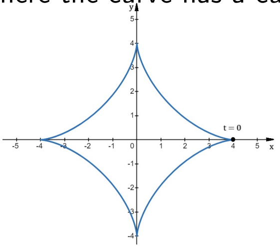
$$\lim_{t\to 0} \frac{dy}{dx} = \lim_{t\to 0} \frac{y'(t)}{x'(t)} = \lim_{t\to 0} \frac{3\cos(t) - 3\cos(3t)}{-3\sin(t) - 3\sin(3t)}.$$

Since $\lim_{t\to 0} (3\cos(t) - 3\cos(3t)) = 0 = \lim_{t\to 0} (-3\sin(t) - 3\sin(3t))$, we deal with a $\frac{0}{0}$ indeterminate form and can apply L'Hospital's rule:

Part 2

$$\lim_{t\to 0}\frac{dy}{dx}=\lim_{t\to 0}\frac{3\cos(t)-3\cos(3t)}{-3\sin(t)-3\sin(3t)}$$
$$=\lim_{t\to 0}\frac{-3\sin(t)+9\sin(3t)}{-3\cos(t)-9\cos(3t)}$$
$$=\frac{-0+0}{-3-9}=\frac{0}{-12}=0.$$

Therefore, when $t=0$, the slope of the tangent line is zero, and hence the tangent line to the hypocycloid is horizontal at the point $(4,0)$, corresponding to $t=0$, where the curve has a cusp.



Finding a Tangent Line

Find the equation of the tangent line to the parametric curve defined by the equations

$$x(t) = t^2 - 3, \quad y(t) = 2t - 1, \quad -3 \leq t \leq 4$$

at the point corresponding to $t = 2$.

Solution: We first calculate $x'(t)$ and $y'(t)$:

$$x'(t) = 2t, \quad y'(t) = 2.$$

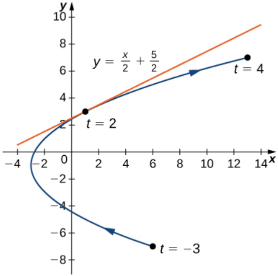
Next we substitute these into (*):

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{2}{2t} = \frac{1}{t}.$$

When $t = 2$, $\frac{dy}{dx} = \frac{1}{2}$, so this is the slope of the tangent line. Calculating $x(2)$ and $y(2)$ gives $x(2) = 2^2 - 3 = 1$ and $y(2) = 2(2) - 1 = 3$, which corresponds to the point $(1, 3)$ on the curve. We now use the point-slope form of the equation of a line to find the equation of the tangent line at this point:

Part 2

$$\begin{aligned} y - y_0 &= m(x - x_0) \\ y - 3 &= \frac{1}{2}(x - 1) \\ y - 3 &= \frac{1}{2}x - \frac{1}{2} \\ y &= \frac{1}{2}x + \frac{5}{2}. \end{aligned}$$



Finding the Equation of the Tangent Line

Find the equation of the tangent line to the curve defined by the equations

$$x(t) = t^2 - 4t, \quad y(t) = 2t^3 - 6t, \quad -2 \leq t \leq 3$$

at the point corresponding to $t = 5$.
Solution: We first calculate $x'(t)$ and $y'(t)$:

$$x'(t) = 2t - 4, \quad y'(t) = 6t^2 - 6.$$

Next, we evaluate $x'(5) = 2(5) - 4 = 6$ and $y'(5) = 6(5)^2 - 6 = 144$. Using the point-slope form of the equation of a line with the point $(x(5), y(5))$ and slope $\frac{dy}{dx}(5)$, we have:

$$\begin{aligned} y - y(5) &= \frac{dy}{dx}(5)(x - x(5)) \\ y - (2(5)^3 - 6(5)) &= \frac{dy}{dx}(5)(x - (5^2 - 4(5))) \\ y - 40 &= \frac{144}{6}(x - 6), \; y - 40 = 24(x - 6), \; y = 24x + 100. \end{aligned}$$

Therefore, the equation of the tangent line is $y = 24x + 100$.

Second Derivative of Parametric Functions

To understand how to take the second derivative of a function defined parametrically, we start by considering the second derivative of a function $y = f(x)$. The second derivative of $y = f(x)$ is defined to be the derivative of the first derivative, which can be represented as:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right].$$

Since $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$, we can replace y on both sides of this equation with $\frac{dy}{dx}$. This substitution leads us to:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}.$$

If we know $\frac{dy}{dx}$ as a function of t, then this formula is straightforward to apply.

Finding a Second Derivative

Calculate the second derivative $\frac{d^2y}{dx^2}$ for the plane curve defined by the parametric equations $x(t) = t^2 - 3, y(t) = 2t - 1$.

Solution:

Using $(*)$, we find that $\frac{dy}{dx} = \frac{2}{2t} = \frac{1}{t}$.

Applying $(**)$, we obtain

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(\frac{1}{t}\right)}{2t} = \frac{-t^{-2}}{2t} = -\frac{1}{2t^3}.$$

Calculating the Second Derivative

Calculate the second derivative $\frac{d^2y}{dx^2}$ for the plane curve defined by the equations

$$x(t) = t^3 + 2t, \quad y(t) = 1 - t + t^2.$$

Solution:

Using the parametric equations, we first find the first derivative $\frac{dy}{dx}$ using the formula $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$.

The first derivatives are: $\frac{dx}{dt} = 3t^2 + 2, \quad \frac{dy}{dt} = -1 + 2t.$

So, the first derivative $\frac{dy}{dx}$ is given by:

$$\frac{dy}{dx} = \frac{-1 + 2t}{3t^2 + 2}.$$

Next, to find the second derivative $\frac{d^2y}{dx^2}$, we differentiate $\frac{dy}{dx}$ with respect to t and then divide by $\frac{dx}{dt}$:

Part 2

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}.$$

Differentiating $\frac{dy}{dx}$, we get:

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{-1+2t}{3t^2+2}\right) = \frac{2(3t^2+2) - 2(-1+2t)(6t)}{(3t^2+2)^2}.$$

So, the second derivative $\frac{d^2y}{dx^2}$ is given by:

$$\frac{d^2y}{dx^2} = \frac{4+6t-12t^2}{(3t^2+2)^3}.$$

Answer:

$$\frac{d^2y}{dx^2} = \frac{4+6t-12t^2}{(3t^2+2)^3}.$$

Examining Concavity of a Parametric Curve

Determine where the parametric curve $x(t) = 4t - t^2$, $y(t) = t^3 + 2$ is concave upward and where it is concave downward.

Solution:

Applying (*), we find that $\frac{dy}{dx} = \frac{3t^2}{4-2t}$. Using (**), together with the quotient rule, we obtain

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\left(\frac{3t^2}{4-2t}\right)'}{4-2t} = \frac{\frac{6t(4-2t)-3t^2(-2)}{(4-2t)^2}}{4-2t} = \frac{\frac{24t-6t^2}{(4-2t)^3}}{4-2t}.$$

We rewrite $\frac{d^2y}{dx^2}$ as

$$\frac{d^2y}{dx^2} = \frac{24t-6t^2}{(4-2t)^3} = \frac{6t(4-t)}{\left(2(2-t)\right)^3} = \frac{6t(4-t)}{2^3(2-t)^3} = \frac{3t(4-t)}{4(2-t)^3}.$$

The numerator has zeros $t = 0$ and $t = 4$, while the denominator has a zero $t = 2$ of multiplicity 3. Using sample points or any other appropriate method, we find that $\frac{d^2y}{dx^2} > 0$, and hence the parametric curve is concave upward, when $t \in (0, 2)$ and $t \in (4, \infty)$, and $\frac{d^2y}{dx^2} < 0$, implying that the curve is concave downward, when $t \in (-\infty, 0)$ and $t \in (2, 4)$.

Concavity of Parametric Curve

Determine where the parametric curve $x(t) = t^2 + 1$, $y(t) = t^2 + t$ is concave upward.

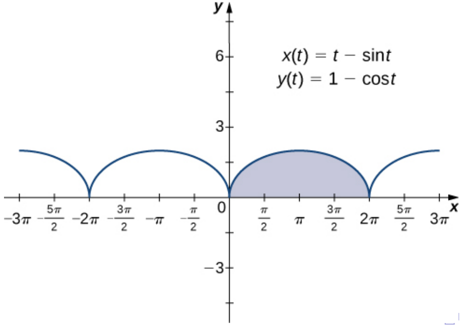
Answer: The curve is concave upward when $t \in (-\infty, 0)$.

Finding the Area under a Parametric Curve

Now that we have seen how to calculate the derivative of a plane curve, the next question is this: How do we find the area under a curve defined parametrically?

Recall the cycloid defined by the equations

$$x(t) = t - \sin(t), \quad y(t) = 1 - \cos(t).$$



Area under a Parametric Curve

Consider the plane curve defined by the parametric equations

$$x = x(t), \quad y = y(t) \geq 0, \quad a \leq t \leq b$$

and assume that $x(t)$ is differentiable.
If $x(t)$ is increasing, then the area under this curve is given by

$$A = \int_a^b y(t)\frac{dx}{dt}\,dt.$$

If $x(t)$ is decreasing, then the area under this curve is given by

$$A = \int_b^a y(t)\frac{dx}{dt}\,dt.$$

Finding the Area under a Parametric Curve

Find the area under one arc of the cycloid defined by the equations

$$x(t) = t - \sin(t), \quad y(t) = 1 - \cos(t), \quad 0 \leq t \leq 2\pi.$$

Solution: To determine whether $x(t)$ is increasing or decreasing, we look at the sign of $x'(t)$. We have that $x'(t) = 1 - \cos(t) \geq 0$, and hence $x(t)$ is increasing. Applying the above theorem, we have

$$\begin{aligned}
A &= \int_a^b y(t)\frac{dx}{dt}\,dt = \int_0^{2\pi} (1-\cos(t))\,(1-\cos(t))\,dt \\
&= \int_0^{2\pi} \left(1 - 2\cos(t) + \cos^2(t)\right)\,dt = \int_0^{2\pi} \left(1 - 2\cos(t) + \frac{1+\cos(2t)}{2}\right)\,dt \\
&= \int_0^{2\pi} \left(\frac{3}{2} - 2\cos(t) + \frac{\cos(2t)}{2}\right)\,dt \\
&= \left(\frac{3t}{2} - 2\sin(t) + \frac{\sin(2t)}{4}\right)\Bigg|_0^{2\pi} \\
&= 3\pi.
\end{aligned}$$

Finding the Area under a Parametric Curve

Find the area under the upper half of the hypocycloid defined by the equations

$$x(t) = 3\cos(t) + \cos(3t), \quad y(t) = 3\sin(t) - \sin(3t), \quad 0 \le t \le \pi.$$

Answer:

$$A = 3\pi$$

Hint: Use the above theorem, along with the identities

$$\sin(\alpha)\sin(\beta) = \frac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)]$$

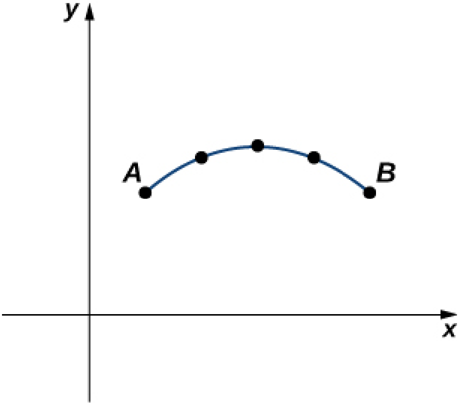
and

$$\sin^2(t) = \frac{1 - \cos(2t)}{2}.$$

Note that $x(t)$ is decreasing.

Approximating the Arc Length of a Parametric Curve

The same way we did for a regular curve with explicit equation $y = f(x)$ or $x = g(y)$, to derive a formula for the arc length of a parametric curve, we approximate it by a union of line segments as shown in the figure above.



Arc Length of a Parametric Curve

Consider the plane curve defined by the parametric equations

$$x = x(t), \quad y = y(t), \quad t_1 \leq t \leq t_2$$

and assume that $x(t)$ and $y(t)$ are smooth, that is, their derivatives $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are continuous. Then the arc length of this curve is given by

$$s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Arc Length Formula for a Regular Curve: Proof

Now suppose that the parameter can be eliminated, leading to a function $y = F(x)$.

$$s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$
$$= \int_{t_1}^{t_2} x'(t)\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dt.$$

Here we have assumed that $x'(t) > 0$, and the case when $x'(t) < 0$ is analogous (the extra minus is going to disappear when the limits of integration are interchanged). Using a substitution $x = x(t)$, we have that $dx = x'(t)\, dt$, and letting $a = x(t_1)$ and $b = x(t_2)$ we obtain the formula

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx,$$

which is exactly the one we had before.

Finding the Arc Length of a Parametric Curve

Find the arc length of the semicircle defined by the equations

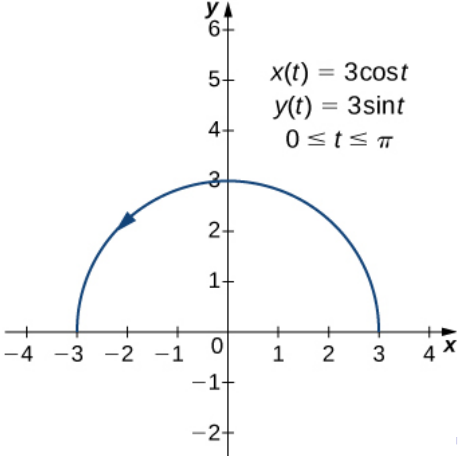
$$x(t) = 3\cos(t), \quad y(t) = 3\sin(t), \quad 0 \le t \le \pi.$$

Solution: The parametric curve is shown in Figure 9 below. To determine its length, we use the formula:

$$\begin{aligned} s &= \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\ &= \int_0^{\pi} \sqrt{(-3\sin(t))^2 + (3\cos(t))^2}\, dt \\ &= \int_0^{\pi} \sqrt{9\sin^2(t) + 9\cos^2(t)}\, dt \\ &= \int_0^{\pi} \sqrt{9(\sin^2(t) + \cos^2(t))}\, dt \\ &= \int_0^{\pi} 3\, dt = 3t\Big|_0^{\pi} = 3\pi. \end{aligned}$$

Note on the Arc Length of a Semicircle

Note that the formula for the arc length of a semicircle is πr, and the radius of this circle is 3. This is a great example of using calculus to derive a known geometric formula.



Finding the Arc Length of a Parametric Curve

Find the arc length of the curve defined by the equations

$$x(t) = 3t^2, \quad y(t) = 2t^3, \quad 1 \leq t \leq 3.$$

Answer:

$$s = 2\left(10^{3/2} - 2^{3/2}\right).$$

Surface Area of a Surface of Revolution

Recall the problem of finding the surface area of a surface of revolution. In Section 2.4, we derived a formula for the surface area of a surface generated by revolving the curve $y = f(x) \geq 0$ from $x = a$ to $x = b$ around the x-axis:

$$S = 2\pi \int_a^b f(x)\sqrt{1 + (f'(x))^2}\, dx.$$

We now consider a surface of revolution generated by revolving a parametrically defined curve $x = x(t),\ y = y(t),\ a \leq t \leq b$ around the x-axis as shown in Figure 11 below.
The formula for its surface area is

$$S = 2\pi \int_a^b y(t)\sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

provided that $y(t)$ is non-negative on $[a, b]$.

Finding Surface Area of a Sphere

Find the surface area of a sphere of radius r centered at the origin.
Solution: We start by parametrizing the upper semicircle with center at the origin and radius r:

$$x(t) = r\cos(t), \quad y(t) = r\sin(t), \quad 0 \leq t \leq \pi.$$

When this curve is revolved around the x-axis, it generates a sphere of radius r. To calculate the surface area of the sphere, we use the formula:

$$\begin{aligned}
S &= 2\pi \int_a^b y(t)\sqrt{(x'(t))^2 + (y'(t))^2}\,dt \\
&= 2\pi \int_0^\pi r\sin(t)\sqrt{(-r\sin(t))^2 + (r\cos(t))^2}\,dt \\
&= 2\pi \int_0^\pi r\sin(t)\sqrt{r^2(\sin^2(t) + \cos^2(t))}\,dt = 2\pi \int_0^\pi r^2\sin(t)\,dt \\
&= 2\pi r^2(-\cos(t))\Big|_0^\pi = 2\pi r^2(-\cos(\pi) + \cos(0)) = 4\pi r^2.
\end{aligned}$$

This agrees with the geometric formula you might have seen before.

Finding the Area of the Surface of Revolution

Find the area of the surface generated by revolving the plane curve defined by the equations

$$x(t) = t^3, \quad y(t) = t^2, \quad 0 \leq t \leq 1$$

around the x-axis.

Answer:

$$A = \frac{\pi(494\sqrt{13} + 128)}{1215}$$

Hint: When evaluating the integral, use a u-substitution.

Key Concepts

- The derivative of the parametrically defined curve $x = x(t)$ and $y = y(t)$ can be calculated using the formula $\frac{dy}{dx} = \frac{y'(t)}{x'(t)}$. Using the derivative, we can find the equation of a tangent line to a parametric curve.
- If $y(t) \geq 0$, the area under the parametric curve can be determined by using the formula $A = \pm \int_a^b y(t)x'(t)\,dt$, where the choice of sign depends on whether $x(t)$ is increasing or decreasing over $[a, b]$.
- The arc length of a parametric curve can be calculated by using the formula $s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$.
- The area of a surface obtained by revolving a parametric curve around the x-axis is given by $S = 2\pi \int_a^b y(t)\sqrt{(x'(t))^2 + (y'(t))^2}\,dt$, provided $y(t) \geq 0$ when $t \in [a, b]$. If the curve is revolved around the y-axis, then the formula is $S = 2\pi \int_a^b x(t)\sqrt{(x'(t))^2 + (y'(t))^2}\,dt$, provided $x(t) \geq 0$ when $t \in [a, b]$.

7.3 Polar Coordinates

Math 1700

University of Manitoba

Winter 2024

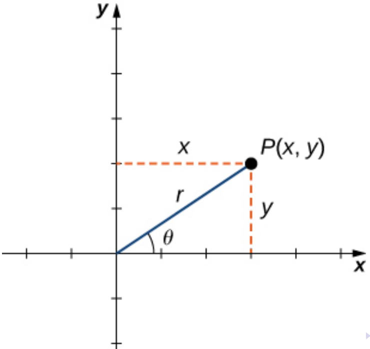
Outline

Learning Objectives

- Locate points in a plane using polar coordinates.
- Convert points between rectangular and polar coordinates.
- Sketch polar curves with given equations.
- Convert equations between rectangular and polar coordinates.
- Identify symmetry in polar curves and equations.

Polar Coordinates

To find the coordinates of a point in the polar coordinate system, consider the Figure below. The point P has Cartesian coordinates (x, y). Consider the line segment connecting the origin to the point P. Its length is equal to the distance from the origin to P and we denote it by r. We also denote the angle between the positive x-axis and the line segment by θ. Then (r, θ) are the polar coordinates of P.



Converting Points between Coordinate Systems

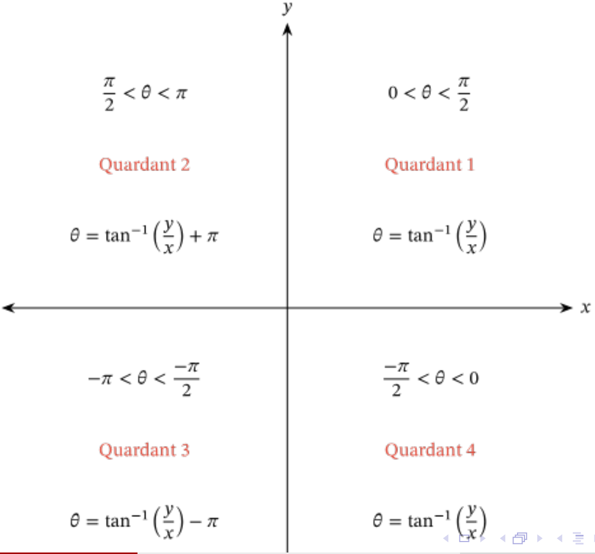
Conversion Formulas

Given a point P in the plane with Cartesian coordinates (x, y) and polar coordinates (r, θ), the following conversion formulas hold true:

(*) $x = r\cos(\theta)$ and $y = r\sin(\theta)$,

(**) $r^2 = x^2 + y^2$ and $\tan(\theta) = \frac{y}{x}$.

Quadrants


y
x
$\frac{\pi}{2} < \theta < \pi$
Quardant 2
$\theta = \tan^{-1}\left(\frac{y}{x}\right) + \pi$
$0 < \theta < \frac{\pi}{2}$
Quardant 1
$\theta = \tan^{-1}\left(\frac{y}{x}\right)$
$-\pi < \theta < \frac{-\pi}{2}$
Quardant 3
$\theta = \tan^{-1}\left(\frac{y}{x}\right) - \pi$
$\frac{-\pi}{2} < \theta < 0$
Quardant 4
$\theta = \tan^{-1}\left(\frac{y}{x}\right)$

Converting Rectangular to Polar Coordinates: Example 1

Convert the rectangular coordinates $(1, 1)$ into polar coordinates:

Converting Rectangular to Polar Coordinates: Example 1

Convert the rectangular coordinates $(1, 1)$ into polar coordinates:
Solution:

$$\begin{aligned} r^2 &= x^2 + y^2 = 1^2 + 1^2 = 2 \\ r &= \sqrt{2} \\ \tan(\theta) &= \frac{y}{x} = 1 \\ \theta &= \frac{\pi}{4} \end{aligned}$$

Therefore, the polar coordinates are $\left(\sqrt{2}, \frac{\pi}{4}\right)$.

Converting Rectangular to Polar Coordinates: Example 2

Convert the rectangular coordinates $(-3, 4)$ into polar coordinates:

Converting Rectangular to Polar Coordinates: Example 2

Convert the rectangular coordinates $(-3, 4)$ into polar coordinates:
Solution:

$$\begin{aligned} r^2 &= x^2 + y^2 = (-3)^2 + 4^2 = 25 \\ r &= 5 \\ \tan(\theta) &= \frac{y}{x} = -\frac{4}{3} \\ \theta &= \pi - \arctan\left(\frac{4}{3}\right) \end{aligned}$$

Therefore, the polar coordinates are $(5, \pi - \arctan\left(\frac{4}{3}\right))$.

Converting Rectangular to Polar Coordinates: Example 3

Convert the rectangular coordinates $(0, 3)$ into polar coordinates:

Converting Rectangular to Polar Coordinates: Example 3

Convert the rectangular coordinates $(0, 3)$ into polar coordinates:
Solution:

$$r = 3$$
$$\theta = \frac{\pi}{2}$$

Therefore, the polar coordinates are $(3, \frac{\pi}{2})$.

Converting Rectangular to Polar Coordinates: Example 4

Convert the rectangular coordinates $(5\sqrt{3}, -5)$ into polar coordinates:

Converting Rectangular to Polar Coordinates: Example 4

Convert the rectangular coordinates $(5\sqrt{3}, -5)$ into polar coordinates: Solution:

$$r = 10$$
$$\theta = -\frac{\pi}{6}$$

Therefore, the polar coordinates are $(10, -\frac{\pi}{6})$.

Converting Polar to Rectangular Coordinates: Example 1

Convert the polar coordinates $\left(3, \frac{\pi}{3}\right)$ into rectangular coordinates:

Converting Polar to Rectangular Coordinates: Example 1

Convert the polar coordinates $\left(3, \frac{\pi}{3}\right)$ into rectangular coordinates:
Solution:

$$x = r\cos(\theta) = 3\cos\left(\frac{\pi}{3}\right) = \frac{3}{2}$$
$$y = r\sin(\theta) = 3\sin\left(\frac{\pi}{3}\right) = \frac{3\sqrt{3}}{2}$$

Therefore, the rectangular coordinates are $\left(\frac{3}{2}, \frac{3\sqrt{3}}{2}\right)$.

Converting Polar to Rectangular Coordinates: Example 2

Convert the polar coordinates $\left(2, \frac{3\pi}{2}\right)$ into rectangular coordinates:

Converting Polar to Rectangular Coordinates: Example 2

Convert the polar coordinates $\left(2, \frac{3\pi}{2}\right)$ into rectangular coordinates:
Solution:

$$x = 0$$
$$y = -2$$

Therefore, the rectangular coordinates are $(0, -2)$.

Converting Polar to Rectangular Coordinates: Example 3

Convert the polar coordinates $\left(6, -\frac{5\pi}{6}\right)$ into rectangular coordinates:

Converting Polar to Rectangular Coordinates: Example 3

Convert the polar coordinates $\left(6, -\frac{5\pi}{6}\right)$ **into rectangular coordinates:**
Solution:

$$x = -3\sqrt{3}$$
$$y = -3$$

Therefore, the rectangular coordinates are $(-3\sqrt{3}, -3)$.

Converting Rectangular to Polar Coordinates

Convert the rectangular coordinates $(-8, -8)$ into polar coordinates:

Converting Rectangular to Polar Coordinates

Convert the rectangular coordinates $(-8, -8)$ into polar coordinates:
Solution:

$$\begin{aligned} r^2 &= x^2 + y^2 = (-8)^2 + (-8)^2 = 128 \\ r &= \sqrt{128} = 8\sqrt{2} \\ \tan(\theta) &= \frac{y}{x} = \frac{-8}{-8} = 1 \\ \theta &= \tan^{-1}(1) - \pi \quad \text{(since in the third quadrant)} \end{aligned}$$

Therefore, the polar coordinates are $\left(8\sqrt{2}, -\frac{3\pi}{4}\right)$.

Converting Polar to Rectangular Coordinates

Convert the polar coordinates $\left(4, \frac{2\pi}{3}\right)$ into rectangular coordinates:

Converting Polar to Rectangular Coordinates

Convert the polar coordinates $\left(4, \frac{2\pi}{3}\right)$ into rectangular coordinates:
Solution:

$$x = r\cos(\theta) = 4\cos\left(\frac{2\pi}{3}\right) = 4 \times \left(-\frac{1}{2}\right) = -2$$
$$y = r\sin(\theta) = 4\sin\left(\frac{2\pi}{3}\right) = 4 \times \frac{\sqrt{3}}{2} = 2\sqrt{3}$$

Therefore, the rectangular coordinates are $\left(-2, 2\sqrt{3}\right)$.

Non-Uniqueness of Polar Representation

Example: The point $(1, \sqrt{3})$ in the rectangular system has multiple polar representations.

For instance:

$$\left(2, \frac{\pi}{3}\right) \text{ and } \left(2, \frac{7\pi}{3}\right)$$

both represent the same point.

Solution: Both polar representations correspond to the same point in the rectangular system.

Usage of Negative Radius in Polar Coordinates

Example: The point $(1, \sqrt{3})$ in the rectangular system can also be represented using negative radius in polar coordinates.
For instance:

$$\left(-2, \frac{4\pi}{3}\right)$$

Solution: Using the conversion formulas:

$$x = r\cos(\theta) = -2\cos\left(\frac{4\pi}{3}\right) = 1$$
$$y = r\sin(\theta) = -2\sin\left(\frac{4\pi}{3}\right) = \sqrt{3}$$

Therefore, $\left(-2, \frac{4\pi}{3}\right)$ represents the point $(1, \sqrt{3})$.

Important

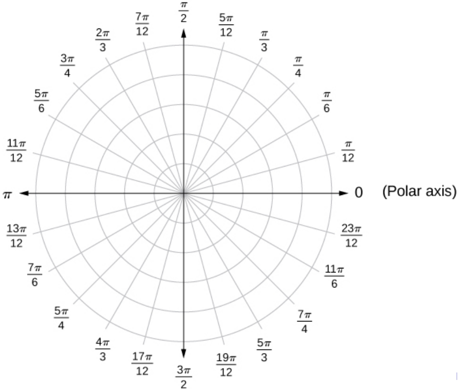
Geometrically, when we plot a point with a negative radial coordinate, we measure the distance of $|r|$ along the halfline that is in the opposite direction to the one that makes the angle of θ with the positive x-axis, so basically the minus reverses the direction, the same way as with angles.)

Infinite number of polar coordinates

Every point in the plane has an infinite number of representations in polar coordinates. However, each point in the plane has only one representation in the rectangular coordinate system.

Polar Coordinate System

- r is the directed distance that the point lies from the origin and θ measures the angle that the line segment from the origin to the point makes with the positive x-axis.
- Positive angles are measured in a counterclockwise direction, and negative angles are measured in a clockwise direction.

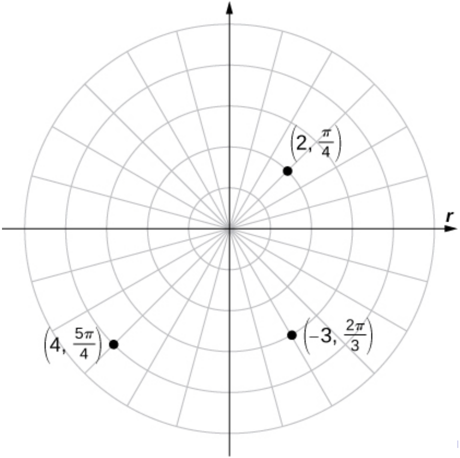


Plotting Points on Polar Plane

Solution

Plot each of the following points on the polar plane:

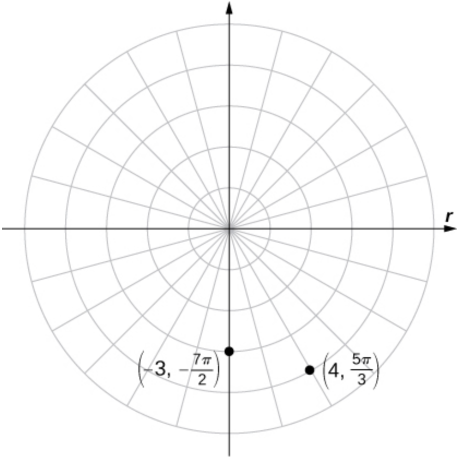
- $(2, \frac{\pi}{4})$, $(-3, \frac{2\pi}{3})$ and $(4, \frac{5\pi}{4})$



Plotting Points on the Polar Plane

Plot the points $\left(4, \frac{5\pi}{3}\right)$ and $\left(-3, -\frac{7\pi}{2}\right)$ on the polar plane.

Solution: The points are plotted in the following figure.



Plotting Curves in the Polar Coordinate System

Now that we know how to plot points in the polar coordinate system, let's discuss how to plot curves.

In the rectangular coordinate system, we can graph a function $y = f(x)$ and create a curve in the Cartesian plane. Similarly, in the polar coordinate system, we can graph a curve that is generated by a function $r = f(\theta)$.

- In this context, r represents the distance from the origin to a point on the curve, and θ represents the angle that the line segment from the origin to that point makes with the positive x-axis.
- To plot a curve given by $r = f(\theta)$, we evaluate r for various values of θ, and then plot the corresponding points in the polar plane.
- Connecting these points with smooth lines or curves gives us the graph of the polar function.

We'll explore this concept further with examples in the upcoming slides.

Problem-Solving Strategy: Plotting a Curve in Polar Coordinates

To plot a curve in polar coordinates, follow these steps:

Five Steps

1. Create a table with two columns. The first column is for θ, and the second column is for r.
2. Create a list of values for θ.
3. Calculate the corresponding r values for each θ.
4. Plot each ordered pair (r, θ) on the coordinate axes.
5. Connect the points and look for a pattern.

This strategy helps in visualizing and understanding the behavior of curves in the polar coordinate system.

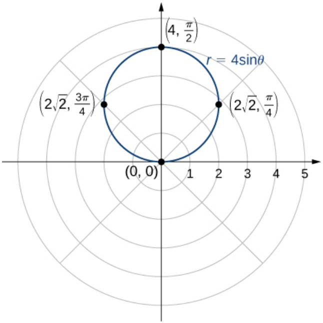
Graphing a Function in Polar Coordinates

Graph the curve defined by the function $r = 4\sin(\theta)$. Identify the curve and rewrite the equation in rectangular coordinates.

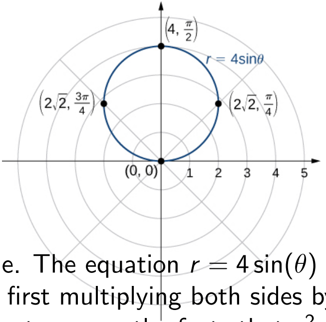
Solution: Because the function is a multiple of a sine function, it is periodic with period 2π. We will use values for θ between 0 and 2π. The result of steps 1–3 appear in the following table:

θ	$r = 4\sin(\theta)$	θ	$r = 4\sin(\theta)$
0	0	π	0
$\frac{\pi}{6}$	2	$\frac{7\pi}{6}$	-2
$\frac{\pi}{4}$	$2\sqrt{2} \approx 2.8$	$\frac{5\pi}{4}$	$-2\sqrt{2} \approx -2.8$
$\frac{\pi}{3}$	$2\sqrt{3} \approx 3.4$	$\frac{4\pi}{3}$	$-2\sqrt{3} \approx -3.4$
$\frac{\pi}{2}$	4	$\frac{3\pi}{2}$	4
$\frac{2\pi}{3}$	$2\sqrt{3} \approx 3.4$	$\frac{5\pi}{3}$	$-2\sqrt{3} \approx -3.4$
$\frac{3\pi}{4}$	$2\sqrt{2} \approx 2.8$	$\frac{7\pi}{4}$	$-2\sqrt{2} \approx -2.8$
$\frac{5\pi}{6}$	2	$\frac{11\pi}{6}$	-2
2π	0		

Graph and center



Graph and center



This is the graph of a circle. The equation $r = 4\sin(\theta)$ can be converted into rectangular coordinates by first multiplying both sides by r. This gives the equation $r^2 = 4r\sin(\theta)$. Next, we use the facts that $r^2 = x^2 + y^2$ and $y = r\sin(\theta)$. This gives $x^2 + y^2 = 4y$. To put this equation into standard form, we subtract $4y$ from both sides of the equation and complete the square:

$$\begin{aligned} x^2 + y^2 - 4y &= 0 \\ x^2 + (y^2 - 4y) &= 0 \\ x^2 + (y^2 - 4y + 4) &= 0 + 4 \\ x^2 + (y - 2)^2 &= 4. \end{aligned}$$

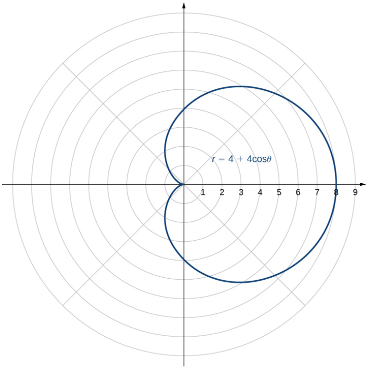
Graph of $r = 4 + 4\cos(\theta)$

Hint: Follow the problem-solving strategy for creating a graph in polar coordinates.

Graph of $r = 4 + 4\cos(\theta)$

Hint: Follow the problem-solving strategy for creating a graph in polar coordinates.

Solution:



The name of this shape is a cardioid, which we will study further later in this section.

Transforming Polar Equations to Rectangular Coordinates

Example 1: Rewrite $\theta = \frac{\pi}{3}$ in rectangular coordinates and identify the graph.

Solution: Take the tangent of both sides. This gives $\tan(\theta) = \tan(\frac{\pi}{3}) = \sqrt{3}$. Since $\tan(\theta) = \frac{y}{x}$, we can replace the left-hand side with $\frac{y}{x}$, resulting in $\frac{y}{x} = \sqrt{3}$. This equation represents a straight line passing through the origin with slope $\sqrt{3}$. Therefore, the graph represents a line passing through the origin with a slope of $\sqrt{3}$. In general, any polar equation of the form $\theta = K$ represents a straight line through the pole with slope equal to $\tan(K)$.

Transforming Polar Equations to Rectangular Coordinates

Example 2: Rewrite $r = 3$ in rectangular coordinates and identify the graph.
Solution: First, square both sides of the equation. This gives $r^2 = 9$. Next, replace r^2 with $x^2 + y^2$, resulting in $x^2 + y^2 = 9$, which is the equation of a circle centered at the origin with radius 3.
In general, any polar equation of the form $r = k$, where k is a constant, represents a circle of radius $|k|$ centered at the origin. (Note: when squaring both sides of an equation, it is possible to introduce new points unintentionally. This should always be taken into consideration. However, in this case, we do not introduce new points. For example, $(-3, \frac{\pi}{3})$ is the same point as $(3, \frac{4\pi}{3})$.)

Transforming Polar Equations to Rectangular Coordinates

Example 3: Rewrite $r = 6\cos(\theta) - 8\sin(\theta)$ in rectangular coordinates and identify the graph.
Solution: Multiplying both sides by r gives $r^2 = 6r\cos(\theta) - 8r\sin(\theta)$. Substituting $x = r\cos(\theta)$ and $y = r\sin(\theta)$, we get $x^2 + y^2 = 6x - 8y$. Completing the square yields $(x-3)^2 + (y+4)^2 = 25$, which is the equation of a circle with center at $(3, -4)$ and radius 5. Notice that the circle passes through the origin since the center is 5 units away.

Rewriting Polar Equation in Rectangular Coordinates

To rewrite the given polar equation $r = \sec(\theta)\tan(\theta)$ in rectangular coordinates.

Rewriting Polar Equation in Rectangular Coordinates

To rewrite the given polar equation $r = \sec(\theta)\tan(\theta)$ in rectangular coordinates.
Solution:
The trigonometric identities we'll use are:

$$\sec(\theta) = \frac{1}{\cos(\theta)}, \ \tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$

Substituting these identities into the equation $r = \sec(\theta)\tan(\theta)$, we get:

$$r = \frac{1}{\cos(\theta)} \cdot \frac{\sin(\theta)}{\cos(\theta)}$$

Now, let's express r in terms of x and y. Since $x = r\cos(\theta)$ and $y = r\sin(\theta)$, we have:

$$r^2 = x^2 + y^2$$

Therefore, our equation becomes: $x^2 + y^2 = \frac{y}{x}$
Multiplying both sides by x to clear the fraction, we obtain:

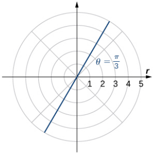
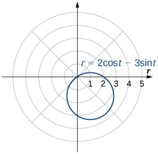
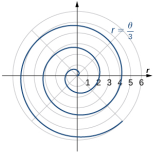
$$x^3 + xy^2 = y$$

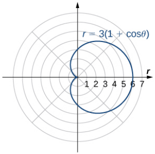
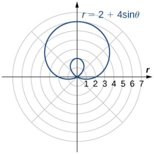
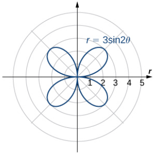
This equation represents a curve in rectangular coordinates. Specifically, it's the

Summary of Common Curves Defined by Polar Equations

Polar Equation	Description
$\theta = K$	Line
$r = a\cos\theta + b\sin\theta$	Circle
$r = a\sin(\theta)$	Circle with radius a centered on x-axis
$r = a\cos(\theta)$	Circle with radius a centered on y-axis
$r = a + b\theta$	Spiral
$r = a \pm b\sin(\theta)$	Cardioid if $a = b$
$r = a \pm b\cos(\theta)$	Cardioid if $a = b$
$r = a + b\sin(\theta)$	Limaçon with a loops if $b > a$
$r = a + b\cos(\theta)$	Limaçon with a loops if $b > a$
$r = a\sin(2\theta)$	Rose with a petals
$r = a\cos(2\theta)$	Rose with a petals

Figures

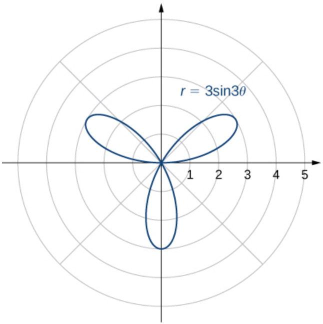
Name	Equation	Example
Line passing through the pole with slope tan K	$\theta = K$	
Circle	$r = a\cos\theta + b\sin\theta$	
Spiral	$r = a + b\theta$	

Name	Equation	Example
Cardioid	$r = a(1 + \cos\theta)$ $r = a(1 - \cos\theta)$ $r = a(1 + \sin\theta)$ $r = a(1 - \sin\theta)$	
Limaçon	$r = a\cos\theta + b$ $r = a\sin\theta + b$	
Rose	$r = a\cos(b\theta)$ $r = a\sin(b\theta)$	

Cardioid and Rose Curves

Cardioid:

- A cardioid is a special case of a limaçon where $a = b$ or $a = -b$.



Rose Curve:

- The graph of $r = 3\sin(2\theta)$ has four petals.
- The graph of $r = 3\sin(3\theta)$ has three petals.
- **If the coefficient is irrational, then the curve never closes,**

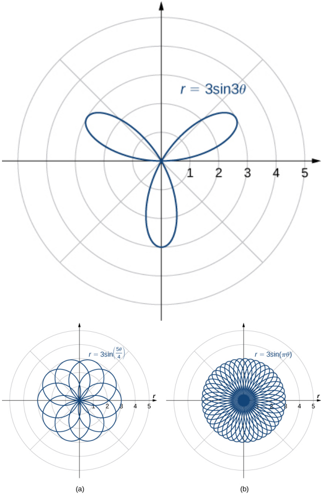
Cardioid and Rose Curves

Cardioid:

- A cardioid is a special case of a limaçon where $a = b$ or $a = -b$.

Rose Curve:

- The graph of $r = 3\sin(2\theta)$ has four petals.
- The graph of $r = 3\sin(3\theta)$ has three petals.
- **If the coefficient is irrational, then the curve never closes,**



Calculus with Polar Curves

Find the slope of the tangent line to the spiral with polar equation $r = \pi - \theta$ at the point corresponding to $\theta = \frac{2\pi}{3}$.

Solution:

$$\begin{aligned} x &= r\cos(\theta) = (\pi - \theta)\cos(\theta) \\ y &= r\sin(\theta) = (\pi - \theta)\sin(\theta) \end{aligned}$$

Next, find $\frac{dy}{dx}$ as a function of θ:

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy/d\theta}{dx/d\theta} \\ &= \frac{\frac{d}{d\theta}\big((\pi - \theta)\sin(\theta)\big)}{\frac{d}{d\theta}\big((\pi - \theta)\cos(\theta)\big)} \\ &= \frac{-\sin(\theta) + (\pi - \theta)\cos(\theta)}{-\cos(\theta) + (\pi - \theta)(-\sin(\theta))} \end{aligned}$$

Part 2

The slope m of the tangent line at $\theta = \frac{2\pi}{3}$ is:

$$\begin{aligned} m &= \frac{dy}{dx}\left(\frac{2\pi}{3}\right) \\ &= \frac{-\sin\left(\frac{2\pi}{3}\right) + \left(\pi - \frac{2\pi}{3}\right)\cos\left(\frac{2\pi}{3}\right)}{-\cos\left(\frac{2\pi}{3}\right) + \left(\pi - \frac{2\pi}{3}\right)\left(-\sin\left(\frac{2\pi}{3}\right)\right)} \\ &= \frac{-\frac{\sqrt{3}}{2} + \frac{\pi}{3}\cdot\left(-\frac{1}{2}\right)}{-\left(-\frac{1}{2}\right) + \frac{\pi}{3}\cdot\left(-\frac{\sqrt{3}}{2}\right)} \\ &= \frac{3\sqrt{3}+\pi}{-3+\sqrt{3}\pi} \end{aligned}$$

Calculus with Polar Curves

Find the slope of the tangent line to the polar curve $r = 1 + \sin(\theta)$ at the point corresponding to $\theta = -\frac{\pi}{4}$.

Solution: To find the slope of the tangent line, we first need to find the derivative of r with respect to θ, denoted as $\frac{dr}{d\theta}$.

Given the polar equation $r = 1 + \sin(\theta)$, we differentiate it with respect to θ using the chain rule:

$$\frac{dr}{d\theta} = \frac{d}{d\theta}(1 + \sin(\theta)) = \cos(\theta)$$

Now, evaluate $\frac{dr}{d\theta}$ at $\theta = -\frac{\pi}{4}$:

$$\left.\frac{dr}{d\theta}\right|_{\theta=-\frac{\pi}{4}} = \cos\left(-\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

The slope of the tangent line at $\theta = -\frac{\pi}{4}$ is the negative reciprocal of $\frac{dr}{d\theta}$:

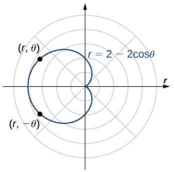
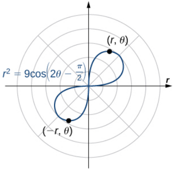
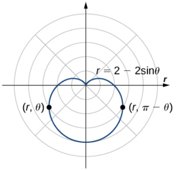
$$\text{Slope} = -\frac{1}{\frac{1}{\sqrt{2}}} = -\sqrt{2} + 1$$

Symmetry in Polar Curves and Equations

Consider a polar curve with equation $r = f(\theta)$.

- The curve is symmetric about the polar axis if for every point (r, θ) on the graph, the point $(r, -\theta)$ is also on the graph. This happens if $f(-\theta) = f(\theta)$ or $f(\pi - \theta) = -f(\theta)$.
- The curve is symmetric about the pole if for every point (r, θ) on the graph, the point $(r, \pi + \theta)$ is also on the graph. This happens if $f(\pi + \theta) = f(\theta)$.
- The curve is symmetric about the vertical line $\theta = \frac{\pi}{2}$ if for every point (r, θ) on the graph, the point $(r, \pi - \theta)$ is also on the graph. This happens if $f(\pi - \theta) = f(\theta)$ or $f(-\theta) = -f(\theta)$.

Examples of each type of symmetry

Symmetry with respect to the polar axis: For every point (r, θ) on the graph, there is also a point reflected directly across the horizontal (polar) axis.	
Symmetry with respect to the pole: For every point (r, θ) on the graph, there is also a point on the graph that is reflected through the pole as well.	
Symmetry with respect to the vertical line $\theta = \frac{\pi}{2}$: For every point (r, θ) on the graph, there is also a point reflected directly across the vertical axis.	

Using Symmetry to Graph a Polar Equation

Determine all symmetries of the rose

The rose is defined by the equation $r = 3\sin(2\theta)$.

Solution

Suppose the point (r, θ) is on the graph of $r = 3\sin(2\theta)$. Let $f(\theta) = 3\sin(2\theta)$. We first substitute $-\theta$ instead of θ into f:

$$f(-\theta) = 3\sin(-2\theta) = -3\sin(2\theta) = -f(\theta)$$

since sine is an odd function. According to iii in the statement above, this implies symmetry with respect to the vertical line $\theta = \frac{\pi}{2}$.
To test for symmetry with respect to the polar axis, we consider $f(\pi - \theta)$:

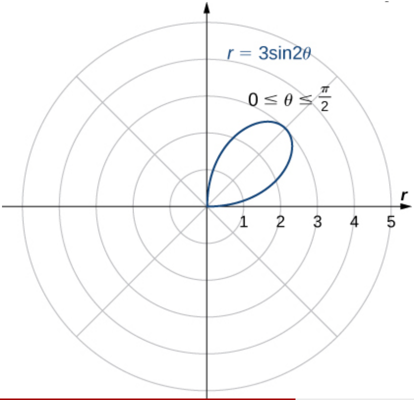
$$f(\pi - \theta) = 3\sin(2\pi - 2\theta) = 3\sin(-2\theta) = -3\sin(2\theta)$$

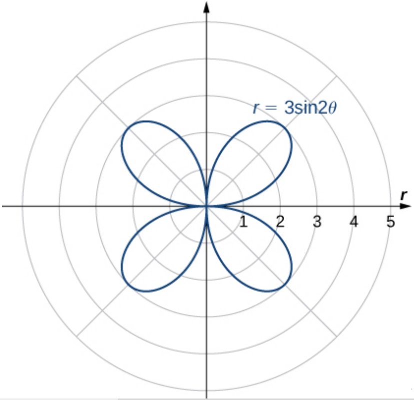
since sine function is 2π-periodic and odd. Hence, by i, we have that the

Graphs : Reflecting into the other three quadrants

Table of Values

θ	r
0	0
$\frac{\pi}{6}$	$\frac{3\sqrt{3}}{2} \approx 2.6$
$\frac{\pi}{4}$	3
$\frac{\pi}{3}$	$\frac{3\sqrt{3}}{2} \approx 2.6$
$\frac{\pi}{2}$	0





Key Concepts

- The polar coordinate system provides an alternative way to locate points in the plane.
- Convert points between rectangular and polar coordinates using the formulas:

$$x = r\cos(\theta)$$
$$y = r\sin(\theta)$$
$$r = \sqrt{x^2 + y^2}$$
$$\tan(\theta) = \frac{y}{x}$$

- To sketch a polar curve, make a table of values and take advantage of periodic properties.
- Use the conversion formulas to convert equations between rectangular and polar coordinates.
- Identify symmetry in polar curves, which can occur through the pole, the horizontal axis, or the vertical axis.

7.4 Area and Arc Length in Polar Coordinates

Math 1700

University of Manitoba

Winter 2024

Outline

Learning Objectives

- Derive the formula for the area of a region in polar coordinates.
- Determine the arc length of a polar curve.

Area and Arc Length in Polar Coordinates

In the rectangular coordinate system, the definite integral provides a way to calculate the area under a curve. In particular, if we have a function $y = f(x)$ defined from $x = a$ to $x = b$ where $f(x) > 0$ on this interval,

Area between the curve and the x-axis

The area between the curve and the x-axis is given by

$$A = \int_a^b f(x)\,dx.$$

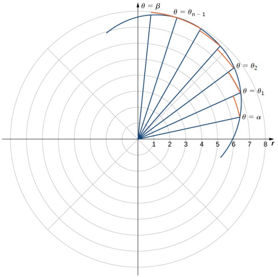
Arc length of this curve

We can also find the arc length of this curve using the formula

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\,dx.$$

Area Bounded by a Polar Curve

Consider a polar curve defined by the function $r = f(\theta)$, where $\alpha \leq \theta \leq \beta$. Our first step is to partition the interval $[\alpha, \beta]$ into n equal-width subintervals. Thus $\Delta\theta = \frac{(\beta-\alpha)}{n}$, and the ith partition point $\theta_i = \alpha + i\Delta\theta$. Each partition point $\theta = \theta_i$ defines a line with slope $\tan(\theta_i)$ passing through the pole as shown in the following graph.



The area of a sector of a circle with angle θ_i can be given as:

$$A_i = \frac{1}{2}(\Delta\theta)\,(f(\theta_i))^2 = \frac{1}{2}(f(\theta_i))^2\Delta\theta.$$

Exact Area Calculation

Summing the areas of sectors for $1 \le i \le n$, we obtain a Riemann sum that approximates the polar area:

$$A \approx \sum_{i=1}^{n} A_i = \sum_{i=1}^{n} \frac{1}{2}(f(\theta_i))^2 \Delta\theta.$$

We take the limit as $n \to \infty$ to get the exact area:

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{2}(f(\theta_i))^2 \Delta\theta = \frac{1}{2}\int_{\alpha}^{\beta} (f(\theta))^2 \, d\theta.$$

Area of a Region Bounded by a Polar Curve

Formula

Suppose f is continuous and nonnegative on the interval $\alpha \leq \theta \leq \beta$ with $0 < \beta - \alpha \leq 2\pi$. The area of the region bounded by the graph of $r = f(\theta)$ between the radial lines $\theta = \alpha$ and $\theta = \beta$ is:

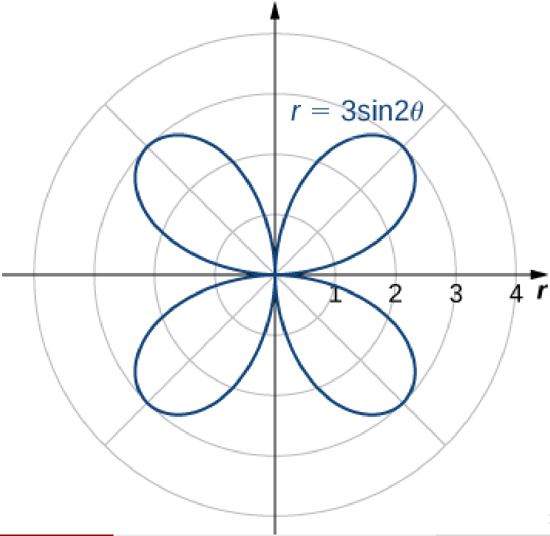
$$(*) \qquad A = \frac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2 \, d\theta = \frac{1}{2}\int_{\alpha}^{\beta} r^2 \, d\theta.$$

Example: Finding the Area of a Polar Region

Find the area of one petal of the rose defined by the equation $r = 3\sin(2\theta)$.

Graph

The graph of $r = 3\sin(2\theta)$ is shown below.



Finding the Area Inside the Petal: Solution

It follows that the petal in the first quadrant corresponds to $\theta \in \left[0, \frac{\pi}{2}\right]$. To find the area inside this petal, use (*) from the above theorem with $f(\theta) = 3\sin(2\theta)$, $\alpha = 0$, and $\beta = \frac{\pi}{2}$:

$$A = \frac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2 d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} [3\sin(2\theta)]^2 d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} 9\sin^2(2\theta) d\theta.$$

To evaluate this integral, use the formula $\sin^2(\alpha) = \frac{1-\cos(2\alpha)}{2}$ with $\alpha = 2\theta$:

$$A = \frac{1}{2}\int_0^{\frac{\pi}{2}} 9\sin^2(2\theta) d\theta = \frac{9}{2}\int_0^{\frac{\pi}{2}} \frac{1-\cos(4\theta)}{2} d\theta = \frac{9}{4}\int_0^{\frac{\pi}{2}} (1-\cos(4\theta)) d\theta$$

$$= \frac{9}{4}\left(\theta - \frac{\sin(4\theta)}{4}\right)\Bigg|_0^{\frac{\pi}{2}} = \frac{9}{4}\left(\frac{\pi}{2} - \frac{\sin(2\pi)}{4}\right) - \frac{9}{4}\left(0 - \frac{\sin(0)}{4}\right) = \frac{9\pi}{8}.$$

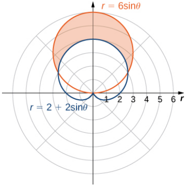
Finding the Area Inside the Cardioid

Problem: Find the area inside the cardioid defined by the equation $r = 1 - \cos(\theta)$.
Answer: $A = \frac{3\pi}{2}$.
Hint: Use (*). Be sure to determine the correct limits of integration before evaluating.

Finding the Area between Two Polar Curves

Problem: Find the area outside the cardioid $r = 2 + 2\sin(\theta)$ and inside the circle $r = 6\sin(\theta)$.

Solution: First draw a graph containing both curves as shown below.



$$6\sin(\theta) = 2 + 2\sin(\theta) \Rightarrow 4\sin(\theta) \quad = 2 \ \Rightarrow \sin(\theta) = \frac{1}{2}.$$

Then $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$ in the interval $(-\pi, \pi]$, which are the limits of integration since from the picture we see that $6\sin(\theta) \geq 2 + 2\sin(\theta)$ on $\left[\frac{\pi}{6}, \frac{5\pi}{6}\right]$. The circle $r = 6\sin(\theta)$ is the red graph, which is the outer function, and the cardioid $r = 2 + 2\sin(\theta)$ is the blue graph, which is the inner function. To calculate the area between the curves, start with the area inside the circle between $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$, then subtract the area inside the cardioid between $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$:

Part 2

$$
\begin{aligned}
A &= \text{circle} - \text{cardioid} \\
&= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} [6\sin(\theta)]^2 d\theta - \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} [2+2\sin(\theta)]^2 d\theta \\
&= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} 36\sin^2(\theta) d\theta - \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (4+8\sin(\theta)+4\sin^2(\theta)) d\theta \\
&= 18\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \frac{1-\cos(2\theta)}{2} d\theta - 2\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (1+2\sin(\theta)+\frac{1-\cos(2\theta)}{2}) d\theta \\
&= 9\left(\theta - \frac{\sin(2\theta)}{2}\right)\Bigg|_{\frac{\pi}{6}}^{\frac{5\pi}{6}} - 2\left(\frac{3\theta}{2} - 2\cos(\theta) - \frac{\sin(2\theta)}{4}\right)\Bigg|_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \\
&= 9\left(\frac{5\pi}{6} - \frac{\sin\left(\frac{5\pi}{3}\right)}{2}\right) - 9\left(\frac{\pi}{6} - \frac{\sin\left(\frac{\pi}{3}\right)}{2}\right) \\
&\quad - \left(3\left(\frac{5\pi}{6}\right) - 4\cos\frac{5\pi}{6} - \frac{\sin\left(\frac{5\pi}{3}\right)}{2}\right) + \left(3\left(\frac{\pi}{6}\right) - 4\cos\frac{\pi}{6} - \frac{\sin\left(\frac{\pi}{3}\right)}{2}\right) = 4\pi.
\end{aligned}
$$

Finding the Area Inside and Outside Circles

Problem: Find the area inside the circle $r = 4\cos(\theta)$ and outside the circle $r = 2$.
Answer: $A = \frac{4\pi}{3} + 2\sqrt{3}$.
Hint: Use (*) and take advantage of symmetry.

Arc Length of a Curve in Polar Coordinates

Here we derive a formula for the arc length of a curve defined in polar coordinates. In rectangular coordinates, the arc length of a parameterized curve $(x(t), y(t))$ for $a \leq t \leq b$ is given by

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

In polar coordinates we define the curve by the equation $r = f(\theta)$, where $\alpha \leq \theta \leq \beta$. In order to adapt the arc length formula for a polar curve, we use the equations

$$x = r\cos(\theta) = f(\theta)\cos(\theta) \quad \text{and} \quad y = r\sin(\theta) = f(\theta)\sin(\theta).$$

Differentiating, we obtain

$$\frac{dx}{d\theta} = f'(\theta)\cos(\theta) - f(\theta)\sin(\theta)$$

$$\frac{dy}{d\theta} = f'(\theta)\sin(\theta) + f(\theta)\cos(\theta).$$

Second part

Applying the known arc length formula, we get

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$= \int_\alpha^\beta \sqrt{\left(f'(\theta)\cos(\theta) - f(\theta)\sin(\theta)\right)^2 + \left(f'(\theta)\sin(\theta) + f(\theta)\cos(\theta)\right)^2}\, d\theta$$

$$= \int_\alpha^\beta \sqrt{(f'(\theta))^2\left(\cos^2(\theta) + \sin^2(\theta)\right) + (f(\theta))^2\left(\cos^2(\theta) + \sin^2(\theta)\right)}\, d\theta$$

$$= \int_\alpha^\beta \sqrt{(f'(\theta))^2 + (f(\theta))^2}\, d\theta = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

Arc Length of a Curve Defined by a Polar Function

Let f be a function whose derivative is continuous on an interval $\alpha \leq \theta \leq \beta$. The length of the polar curve $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ is

Formula

$$L = \int_{\alpha}^{\beta} \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

Finding the Arc Length of a Polar Curve

Problem: Find the arc length of the cardioid $r = 2 + 2\cos(\theta)$.
Solution:

$$\begin{aligned}
L &= \int_{-\pi}^{\pi} \sqrt{[2+2\cos(\theta)]^2 + [-2\sin(\theta)]^2}\, d\theta \\
&= \int_{-\pi}^{\pi} \sqrt{4 + 8\cos(\theta) + 4\cos^2(\theta) + 4\sin^2(\theta)}\, d\theta \\
&= \int_{-\pi}^{\pi} \sqrt{8 + 8\cos(\theta)}\, d\theta \\
&= 2\int_{-\pi}^{\pi} \sqrt{2 + 2\cos(\theta)}\, d\theta = 2\int_{-\pi}^{\pi} \sqrt{4\cos^2\left(\frac{\theta}{2}\right)}\, d\theta \\
&= 2\int_{-\pi}^{\pi} 2\left|\cos\left(\frac{\theta}{2}\right)\right|\, d\theta = 4\int_{-\pi}^{\pi} \cos\left(\frac{\theta}{2}\right)\, d\theta = 4\left(2\sin\left(\frac{\theta}{2}\right)\right)\Bigg|_{-\pi}^{\pi} \\
&= 8(1-(-1)) = 16.
\end{aligned}$$

Finding the Arc Length of $r = 3\sin(\theta)$

Problem: Find the total arc length of $r = 3\sin(\theta)$.
Answer: 3π
Hint To determine the correct limits, make a table of values.
Solution: To determine the correct limits, make a table of values for θ and r, then observe the behavior of r as θ varies.

θ	r
0	0
$\pi/2$	3
π	0
$3\pi/2$	-3
2π	0

As θ goes from 0 to 2π, the curve traces out a single wave of the sine function from $r = 0$ to $r = 3$ and back to $r = 0$. Hence, the total arc length is $s = 3\pi$.

Key Concepts

- The area of the region bounded by the polar curve $r = f(\theta)$ and between the radial lines $\theta = \alpha$ and $\theta = \beta$ is given by the integral

$$A = \frac{1}{2} \int_{\alpha}^{\beta} [f(\theta)]^2 \, d\theta.$$

- To find the area between two curves in the polar coordinate system, first find the points of intersection, then subtract the corresponding areas.
- The arc length of a polar curve defined by the equation $r = f(\theta)$ with $\alpha \le \theta \le \beta$ is given by the integral

$$L = \int_{\alpha}^{\beta} \sqrt{[f(\theta)]^2 + \left[\frac{df}{d\theta}\right]^2} \, d\theta = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta.$$

Key Equations

Area of a region bounded by a polar curve:

$$A = \frac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2 \, d\theta = \frac{1}{2}\int_{\alpha}^{\beta} r^2 \, d\theta$$

Arc length of a polar curve:

$$L = \int_{\alpha}^{\beta} \sqrt{[f(\theta)]^2 + \left[\frac{df}{d\theta}\right]^2} \, d\theta = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta$$