Chapter 3: Techniques of Integration
3.1 Integration by Parts

Math 1700

University of Manitoba

2024

Outline

Introduction

In urban landscapes, strategic traffic signal planning can prevent accidents at busy intersections. Consider a city where changes to traffic lights were made at a problematic junction, resulting in no accidents over eight months. Were these changes effective or coincidental? Integration plays a vital role in answering this question.

Integration is pivotal across disciplines, from computing volumes to pinpointing centers of mass. This chapter delves into advanced integration techniques such as integration by parts and trigonometric integrals, essential for various fields.

Learning Objectives

- Recognize when to use integration by parts.
- Use the integration-by-parts formula to evaluate indefinite integrals.
- Apply the integration-by-parts formula for definite integrals.

Integration by Parts

Let $u = f(x)$ and $v = g(x)$ be functions with continuous derivatives. Then, the integration-by-parts formula for the integral involving these two functions is:

Formula of Integration by Parts

$$\int u\,dv = uv - \int v\,du.$$

Using Integration by Parts

Use integration by parts with $u = x$ and $dv = \sin(x)\,dx$ to evaluate $\int x\sin(x)\,dx$.

Solution: By choosing $u = x$, we have $du = 1\,dx$. Since $dv = \sin(x)\,dx$, we can take v to be any antiderivative of $\sin(x)$, and the simplest choice would be $v = -\cos(x)$. It is handy to keep track of these values as follows:

$$\begin{array}{rclcrl} u & = & x & \quad & dv & = \sin(x)\,dx \\ du & = & dx & \quad & v & = -\cos(x). \end{array}$$

Applying the integration-by-parts formula (*) results in

$$\begin{aligned} \int x\sin(x)\,dx &= (x)(-\cos(x)) - \int(-\cos(x))(dx) \quad \text{(substitute)} \\ &= -x\cos(x) + \int \cos(x)\,dx \quad \text{(simplify)} \\ &= -x\cos(x) + \sin(x) + C. \quad \text{(integrate}\cos(x)\text{)} \end{aligned}$$

Analysis:

Analysis

At this point, there are probably a few items that need clarification. First of all, you may be curious about what would have happened if we had chosen $u = \sin(x)$ and $dv = x$. If we had done so, then we would have $du = \cos(x)$ and $v = \frac{1}{2}x^2$. Thus, after applying integration by parts, we would get

$$\int x \sin(x)\, dx = \frac{1}{2}x^2 \sin(x) - \int \frac{1}{2}x^2 \cos(x)\, dx.$$

Unfortunately, with the new integral, we are in no better position than before. It is important to keep in mind that when we apply integration by parts, we may need to try several choices for u and dv before finding a choice that works.

Second, you may wonder why, when we find v as an antiderivative of $\sin(x)$ we do not use $v = -\cos(x) + K$. To see that it makes no difference, we can rework the problem using $v = -\cos(x) + K$:

Analysis (cont'd)

$$\begin{aligned}
& \int x\sin(x)\,dx \\
= \;& (x)(-\cos(x)+K) - \int(-\cos(x)+K)\,dx \quad \text{(substitute)} \\
= \;& -x\cos(x) + Kx + \int \cos(x)\,dx \quad \text{(simplify)} \\
= \;& -x\cos(x) + \sin(x) + C. \quad \text{(integrate } \cos(x))
\end{aligned}$$

As you can see, it makes no difference in the final solution.
Last, we can verify that our antiderivative is correct by differentiating $-x\cos(x)+\sin(x)+C$:

$$\begin{aligned}
& \frac{d}{dx} \; (-x\cos(x)+\sin(x)+C) \\
= \;& (-1)\cos(x) + (-x)(-\sin(x)) + \cos(x) \\
= \;& x\sin(x).
\end{aligned}$$

Therefore, the answer we obtained is correct.

Example

To evaluate $\int xe^{2x}\,dx$ using integration by parts, we choose $u = x$ and $dv = e^{2x}\,dx$. Then, we have $du = dx$ and $v = \frac{1}{2}e^{2x}$.
Applying the integration-by-parts formula $\int u\,dv = uv - \int v\,du$, we get:

$$\begin{aligned}\int xe^{2x}\,dx &= x \cdot \frac{1}{2}e^{2x} - \int \frac{1}{2}e^{2x}\,dx \\ &= \frac{1}{2}xe^{2x} - \frac{1}{2}\int e^{2x}\,dx \\ &= \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C,\end{aligned}$$

where C is the constant of integration. Therefore, the solution is $\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C$.

Evaluate $\int \frac{\ln(x)}{x^3}\,dx$

Solution:
Begin by rewriting the integral:

$$\int \frac{\ln(x)}{x^3}\,dx = \int x^{-3}\ln(x)\,dx.$$

Since this integral contains the algebraic function x^{-3} and the logarithmic function $\ln(x)$, choose $u = \ln(x)$, since L comes before A in LIATE. After we have chosen $u = \ln(x)$, we must choose $dv = x^{-3}\,dx$.
Next, since $u = \ln(x)$, we have $du = \frac{1}{x}\,dx$. Also, $\int x^{-3}\,dx = -\frac{1}{2}x^{-2} + C$, and so we take $v = -\frac{1}{2}x^{-2}$. Summarizing,

$$\begin{array}{rclcrcl} u &=& \ln(x) & \quad & dv &=& x^{-3}\,dx \\ du &=& \frac{1}{x}\,dx & & v &=& -\frac{1}{2}x^{-2}. \end{array}$$

Substituting into the integration-by-parts formula gives:

$$\begin{aligned} \int \frac{\ln(x)}{x^3}\,dx &= \int x^{-3}\ln(x)\,dx = (\ln(x))\left(-\tfrac{1}{2}x^{-2}\right) - \int\left(-\tfrac{1}{2}x^{-2}\right)\left(\tfrac{1}{x}\,dx\right) \\ &= -\tfrac{1}{2}x^{-2}\ln(x) + \int \tfrac{1}{2}x^{-3}\,dx = -\tfrac{1}{2}x^{-2}\ln(x) - \tfrac{1}{4}x^{-2} + C. \end{aligned}$$

Evaluate $\int x \ln(x)\, dx$

Answer: To evaluate the integral $\int x \ln(x)\, dx$, we will use the integration by parts formula:

$$\int u\, dv = uv - \int v\, du$$

Evaluate $\int x \ln(x)\, dx$

Answer: To evaluate the integral $\int x \ln(x)\, dx$, we will use the integration by parts formula:

$$\int u\, dv = uv - \int v\, du$$

where $u = \ln(x)$ and $dv = x\, dx$.

Solution: We start by choosing $u = \ln(x)$ and $dv = x\, dx$. Then, we find du and v:

$$du = \frac{1}{x}\, dx \quad \text{and} \quad v = \frac{1}{2}x^2$$

Applying the integration by parts formula, we have:

$$\begin{aligned}\int x \ln(x)\, dx &= uv - \int v\, du = \ln(x) \cdot \frac{1}{2}x^2 - \int \frac{1}{2}x^2 \cdot \frac{1}{x}\, dx \\ &= \frac{1}{2}x^2 \ln(x) - \frac{1}{2}\int x\, dx = \frac{1}{2}x^2 \ln(x) - \frac{1}{4}x^2 + C\end{aligned}$$

where C is the constant of integration.

Evaluate $\int x^2 e^{3x}\, dx$

Solution: Using LIATE, choose $u = x^2$ and $dv = e^{3x}\, dx$. Thus, $du = 2x\, dx$ and $\int e^{3x}\, dx = \frac{1}{3}e^{3x} + C$, which means we can take $v = \frac{1}{3}e^{3x}$. Therefore,

$$u = x^2 \qquad dv = e^{3x}\, dx$$
$$du = 2x\, dx \qquad v = \frac{1}{3}e^{3x}$$

Substituting into the integration by parts formula (*), we get

$$\int x^2 e^{3x}\, dx = \frac{1}{3}x^2 e^{3x} - \int \frac{2}{3}xe^{3x}\, dx.$$

We still cannot integrate $\int \frac{2}{3}xe^{3x}\, dx$ directly, but the integral now has a lower power on x. We can evaluate this new integral by using integration by parts again. To do this, choose $u = x$ and $dv = \frac{2}{3}e^{3x}\, dx$. Thus, $du = dx$ and $\int \frac{2}{3}e^{3x}\, dx = \frac{2}{9}e^{3x}$. Now we have

Applying Integration by Parts More Than Once (Continued)

$$u = x \qquad dv = \frac{2}{3}e^{3x}\,dx$$
$$du = dx \qquad v = \frac{2}{9}e^{3x}$$

Going back to the previous equation and using (*), we get

$$\begin{aligned}\int x^2e^{3x}\,dx &= \frac{1}{3}x^2e^{3x} - \int \frac{2}{3}xe^{3x}\,dx \\ &= \frac{1}{3}x^2e^{3x} - \left(\frac{2}{9}xe^{3x} - \int \frac{2}{9}e^{3x}\,dx\right).\end{aligned}$$

After evaluating the last integral and simplifying, we obtain

$$\int x^2e^{3x}\,dx = \frac{1}{3}x^2e^{3x} - \frac{2}{9}xe^{3x} + \frac{2}{27}e^{3x} + C.$$

where C is the constant of integration.

Applying Integration by Parts When LIATE Doesn't Quite Work (Part 1)

Evaluate $\int t^3 e^{t^2}\, dt$.

Solution: If we use a strict interpretation of the mnemonic LIATE to make our choice of u, we end up with $u = t^3$ and $dv = e^{t^2} dt$. Unfortunately, this choice won't work because we are unable to evaluate $\int e^{t^2} dt$. However, since we can evaluate $\int te^{t^2}\, dt$, we can try choosing $u = t^2$ and $dv = te^{t^2} dt$. We then have

$$\begin{aligned} u &= t^2 & dv &= te^{t^2} dt \\ du &= 2t\, dt & v &= \tfrac{1}{2} e^{t^2}. \end{aligned}$$

$$\int t^3 e^{t^2} dt = t^2 \cdot \frac{1}{2} e^{t^2} - \int \frac{1}{2} e^{t^2} 2t\, dt$$

Let us compute $\int te^{t^2}\, dt$

Applying Integration by Parts When LIATE Doesn't Quite Work (Part 2)

$$\begin{aligned}\int te^{t^2}\,dt &= \int e^{t^2}\frac{1}{2}(2t)\,dt = \frac{1}{2}\int e^{t^2}(t^2)'\,dt\\ &= \frac{1}{2}\int e^w\,dw\\ &= \frac{1}{2}e^w + C\\ &= \frac{1}{2}e^{t^2} + C,\end{aligned}$$

Thus, we obtain

$$\int t^3e^{t^2}dt = \frac{1}{2}t^2e^{t^2} - \frac{1}{2}e^{t^2} + C.$$

Applying Integration by Parts More Than Once (Part 1)

Evaluate $\int \sin(\ln(x))\,dx$.

Solution: This integral appears to have only one function—namely, $\sin(\ln(x))$—however, we can always use the constant function 1 as the other function. In this example, let's choose $u = \sin(\ln(x))$ and $dv = 1\,dx$. (The decision to use $u = \sin(\ln(x))$ is easy. We can't choose $dv = \sin(\ln(x))\,dx$ because if we could integrate it, we wouldn't be using integration by parts in the first place!) Consequently, $du = \cos(\ln(x))\left(\frac{1}{x}\right)\,dx$ and we can take $v = x$ as an antiderivative of 1. After applying integration by parts to the integral and simplifying, we obtain

$$\int \sin(\ln(x))\,dx = x\sin(\ln(x)) - \int \cos(\ln(x))\,dx$$

Unfortunately, this process leaves us with a new integral that is very similar to the original.

Applying Integration by Parts More Than Once (Part 2)

However, let's see what happens when we apply integration by parts again. This time let's choose $u = \cos(\ln(x))$ and $dv = 1\,dx$, making $du = -\sin(\ln(x))\left(\frac{1}{x}\right)\,dx$ and, again, $v = x$. Substituting, we have

$$\begin{aligned}\int \sin(\ln(x))\,dx &= x\sin(\ln(x)) - \left(x\cos(\ln(x)) - \int -\sin(\ln(x))\,dx\right) \\ &= x\sin(\ln(x)) - x\cos(\ln(x)) + \int \sin(\ln(x))\,dx.\end{aligned}$$

Substituting I instead of $\int \sin(\ln(x))\,dx$ into the above equality, we obtain:

$$I = x\sin(\ln(x)) - x\cos(\ln(x)) - I.$$

To find I, add I to both sides of the equation:

$$2I = x\sin(\ln(x)) - x\cos(\ln(x)),$$

and then divide by 2: $I = \frac{1}{2}x\sin(\ln(x)) - \frac{1}{2}x\cos(\ln(x))$. Since I was a particular antiderivative of $\sin(\ln(x))$, we have

$$\int \sin(\ln(x))\,dx = I + C = \frac{1}{2}x\sin(\ln(x)) - \frac{1}{2}x\cos(\ln(x)) + C,$$

Applying Integration by Parts

Evaluate $\int x^2 \sin(x)\,dx$.
Solution: Let's use integration by parts with $u = x^2$ and $dv = \sin(x)\,dx$. Then, $du = 2x\,dx$ and $v = -\cos(x)$. Applying the integration by parts formula:

$$\int x^2 \sin(x)\,dx = x^2(-\cos(x)) - \int -\cos(x)\cdot 2x\,dx$$
$$= -x^2\cos(x) + 2\int x\cos(x)\,dx.$$

Now, let's integrate $\int x\cos(x)\,dx$ by parts again. Choosing $u = x$ and $dv = \cos(x)\,dx$, we get $du = dx$ and $v = \sin(x)$. Thus,

$$\int x^2 \sin(x)\,dx = -x^2\cos(x) + 2(x\sin(x) - \int \sin(x)\,dx)$$
$$= -x^2\cos(x) + 2x\sin(x) + 2\cos(x) + C,$$

where C is the constant of integration.

Integration by Parts for Definite Integrals

Integration by Parts Formula for Definite Integrals

Let $f(x)$ and $g(x)$ be functions with continuous derivatives on $[a, b]$. Then

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big|_a^b - \int_a^b g(x)f'(x)\,dx.$$

If we denote $u = f(x)$ and $v = g(x)$, then it becomes

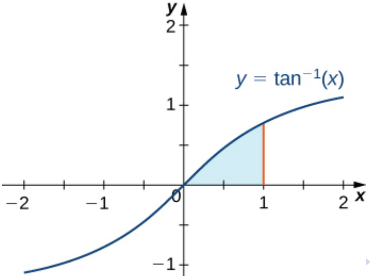
$$(**) \quad \int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du,$$

where the bounds of integration and substitution are specified for the variable x.

Finding the Area of a Region

Problem: Find the area of the region bounded above by the graph of $y = \arctan(x)$ and below by the x-axis over the interval $[0, 1]$.

1. This region is shown in Figure. To find the area, we must evaluate $\int_0^1 \arctan(x)\, dx$.
2. This figure is the graph of the inverse tangent function. It is an increasing function that passes through the origin. In the first quadrant, there is a shaded region under the graph, above the x-axis. The shaded area is bounded to the right at $x = 1$.



Finding the Area of a Region (cont'd)

3. For this integral, let's choose $u = \arctan(x)$ and $dv = dx$, thereby making $du = \frac{1}{x^2+1}dx$ and $v = x$. After applying the integration-by-parts formula (**) for definite integrals, we obtain

$$\text{Area} = x\arctan(x)\Big|_0^1 - \int_0^1 \frac{x}{x^2+1}\,dx.$$

4. We use a substitution of $w = 1 + x^2$ to evaluate $\int_0^1 \frac{x}{x^2+1}\,dx$. We have that $dw = 2x\,dx$, and hence $x\,dx = \frac{1}{2}dw$. Also, when $x = 0$, $w = 1$, and when $x = 1$, $w = 2$. It follows that

$$\int_0^1 \frac{x}{x^2+1}\,dx = \int_1^2 \frac{1}{2}\frac{1}{w}\,dw = \frac{1}{2}\ln|w|\Big|_1^2 = \frac{1}{2}\ln(2).$$

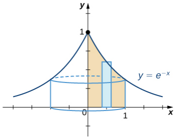
5. Therefore,

$$\text{Area} = x\arctan(x)\Big|_0^1 - \int_0^1 \frac{x}{x^2+1}\,dx = \frac{\pi}{4} - \frac{1}{2}\ln(2).$$

Finding the Volume of Revolution

Problem: Find the volume of the solid obtained by revolving the region bounded by the graph of $f(x) = e^{-x}$, the x-axis, the y-axis, and the line $x = 1$ about the y-axis.

1. We use the cylindrical shells method to solve this problem. Begin by sketching the region to be revolved, along with a typical rectangle.
2. This figure is the graph of the function e^{-x}. It is an increasing function on the left side of the y-axis and decreasing on the right side of the y-axis. The curve also comes to a point on the y-axis at $y = 1$. Under the curve, there is a shaded rectangle in the first quadrant. There is also a cylinder under the graph, formed by revolving the rectangle around the y-axis.



Finding the Volume of Revolution (cont'd)

3. **Figure** We can use cylindrical shells to find the volume of revolution.
4. According to the formula, we must evaluate

$$\int_0^1 2\pi x e^{-x}\,dx = 2\pi \int_0^1 x e^{-x}\,dx.$$

5. To do this, let $u = x$ and $dv = e^{-x}\,dx$. These choices lead to $du = dx$ and $v = -e^{-x}$ as an antiderivative of e^{-x}. Using integration by parts, we obtain

$$\begin{aligned} \text{Volume} = 2\pi \int_0^1 x e^{-x}\,dx &= 2\pi \left(-x e^{-x} \Big|_0^1 + \int_0^1 e^{-x}\,dx \right) \\ &= -2\pi x e^{-x} \Big|_0^1 - 2\pi e^{-x} \Big|_0^1 \\ &= 2\pi - \frac{4\pi}{e}. \end{aligned}$$

Evaluation of $\int_0^{\frac{\pi}{2}} x\cos(x)\,dx$

Solution: Using integration by parts with $u = x$ and $dv = \cos(x)\,dx$, we get:

$$du = dx \quad v = \sin(x)$$

Applying the integration by parts formula,

$$\int u\,dv = uv - \int v\,du,$$

we obtain:

$$[x\sin(x)]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin(x)\,dx = \left(\frac{\pi}{2}\cdot 1 - 0\cdot 0\right) + (0 - (-1)) = \frac{\pi}{2} - 1.$$

Therefore,

$$\int_0^{\frac{\pi}{2}} x\cos(x)\,dx = \frac{\pi}{2} - 1.$$

Key Concepts and Key Equations

- The integration-by-parts formula allows the exchange of one integral for another, possibly easier, integral.
- Integration by parts applies to both definite and indefinite integrals.

Key Equations

Integration by Parts Formula

$$\int u\,dv = uv - \int v\,du$$

Integration by Parts for Definite Integrals

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

3.2 Trigonometric Integrals

Math 1700

University of Manitoba

Winter 2024

Outline

Learning Objectives

1. Solve integration problems involving products of powers of $\sin(x)$ and $\cos(x)$.
2. Integrate products of sines and cosines of different angles.
3. Solve integration problems involving products of powers of $\tan(x)$ and $\sec(x)$.
4. Use reduction formulas to evaluate trigonometric integrals.

A key idea behind the strategy used to integrate combinations of powers of $\sin(x)$ and $\cos(x)$ involves rewriting these expressions as sums and differences of integrals of the form $\int \sin^j(x)\cos(x)\,dx$ or $\int \cos^j(x)\sin(x)\,dx$ that can be evaluated using u-substitution.

Evaluate $\int \cos^3(x)\sin(x)\,dx.$

Solution:

Make a substitution $u = \cos(x)$. In this case, $du = -\sin(x)\,dx$. Thus,

$$\begin{aligned}\int \cos^3(x)\sin(x)\,dx &= -\int u^3\,du \\ &= -\frac{1}{4}u^4 + C \\ &= -\frac{1}{4}\cos^4(x) + C.\end{aligned}$$

Evaluate $\int \sin^4(x)\cos(x)\,dx$.

Answer: $\frac{1}{5}\sin^5(x) + C$

Hint: Take $u = \sin(x)$.

A Preliminary Example: Evaluating $\int \cos^j(x)\sin^k(x)\,dx$ When k is Odd

Evaluate $\int \cos^2(x)\sin^3(x)\,dx$.

Solution: To convert this integral into a combination of integrals of the form $\int \cos^j(x)\sin(x)\,dx$, rewrite

$\sin^3(x) = \sin^2(x)\sin(x) = (1-\cos^2(x))\sin(x)$.

We now make a substitution $u = \cos(x)$, $du = -\sin(x)\,dx$, and obtain

$$\begin{aligned}\int \cos^2(x)\sin^3(x)\,dx &= \int \cos^2(x)(1-\cos^2(x))\sin(x)\,dx \\ &= -\int u^2(1-u^2)\,du = \int (u^4-u^2)\,du \\ &= \frac{1}{5}u^5 - \frac{1}{3}u^3 + C = \frac{1}{5}\cos^5(x) - \frac{1}{3}\cos^3(x) + C.\end{aligned}$$

Development of the Integral

Given the integral $\int \cos^3(x)\sin^2(x)\,dx$, we rewrite $\cos^3(x)$ as $\cos^2(x)\cos(x)$. Then, using the identity $\cos^2(x) = 1 - \sin^2(x)$, we get:

$$\begin{aligned}\int \cos^3(x)\sin^2(x)\,dx &= \int (1-\sin^2(x))\cos(x)\sin^2(x)\,dx \\ &= \int (\sin^2(x) - \sin^4(x))\cos(x)\,dx.\end{aligned}$$

Now, let's make the substitution $u = \sin(x)$. Then, $du = \cos(x)\,dx$. we have:

$$\begin{aligned}\int \cos^3(x)\sin^2(x)\,dx = \int (u^2 - u^4)\,du &= \frac{1}{3}u^3 - \frac{1}{5}u^5 + C \\ &= \frac{1}{3}\sin^3(x) - \frac{1}{5}\sin^5(x) + C.\end{aligned}$$

So, the evaluated integral is $\frac{1}{3}\sin^3(x) - \frac{1}{5}\sin^5(x) + C$.

Integrating an Even Power of $\sin(x)$

Evaluate $\int \sin^2(x)\,dx$.

Solution: To evaluate this integral, let's use the trigonometric identity $\sin^2(x) = \frac{1}{2} - \frac{1}{2}\cos(2x)$. Thus,

$$\begin{aligned}\int \sin^2(x)\,dx &= \int \left(\frac{1}{2} - \frac{1}{2}\cos(2x)\right)\,dx \\ &= \frac{1}{2}x - \frac{1}{4}\sin(2x) + C.\end{aligned}$$

Development of the Integral

Given the integral $\int \cos^2(x)\,dx$, we can use the trigonometric identity $\cos^2(x) = \frac{1}{2} + \frac{1}{2}\cos(2x)$. Thus, we have:

$$\begin{aligned}\int \cos^2(x)\,dx &= \int \left(\frac{1}{2} + \frac{1}{2}\cos(2x)\right)\,dx \\ &= \frac{1}{2}\int dx + \frac{1}{2}\int \cos(2x)\,dx \\ &= \frac{1}{2}x + \frac{1}{4}\sin(2x) + C.\end{aligned}$$

So, the evaluated integral is $\frac{1}{2}x + \frac{1}{4}\sin(2x) + C$.

Problem-Solving Strategy: Integrating Products of Powers of $\sin(x)$ and $\cos(x)$

To evaluate $\displaystyle\int \cos^j(x)\sin^k(x)\,dx$, use the following strategies:

1. If k is odd, rewrite $\sin^k(x) = \sin^{k-1}(x)\sin(x)$ and use the identity $\sin^2(x) = 1 - \cos^2(x)$ to rewrite $\sin^{k-1}(x)$ in terms of $\cos(x)$. Integrate using the substitution $u = \cos(x)$. This substitution makes $du = -\sin(x)\,dx$.
2. If j is odd, rewrite $\cos^j(x) = \cos^{j-1}(x)\cos(x)$ and use the identity $\cos^2(x) = 1 - \sin^2(x)$ to rewrite $\cos^{j-1}(x)$ in terms of $\sin(x)$. Integrate using the substitution $u = \sin(x)$. This substitution makes $du = \cos(x)\,dx$.
3. If both j and k are even, use identities $\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$ and $\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$. After applying these formulas, simplify and reapply strategies 2 and 3 to the combination of powers of $\cos(2x)$ as appropriate.

(Note: If both j and k are odd, either strategy 1 or strategy 2 may be used.)

Evaluating $\int \cos^j(x)\sin^k(x)\,dx$ When j is Odd

Evaluate $A = \int \cos^5(x)\sin^8(x)\,dx$.

Solution: Since the power on $\cos(x)$ is odd, use strategy 2.

$A = \int \cos^4(x)\sin^8(x)\cos(x)\,dx$	Break off $\cos(x)$.
$= \int (\cos^2(x))^2\sin^8(x)\cos(x)\,dx$	Rewrite $\cos^4(x) = (\cos^2(x))^2$.
$= \int (1-\sin^2(x))^2\sin^8(x)\cos(x)\,dx$	Substitute $\cos^2(x) = 1-\sin^2(x)$.
$= \int (1-u^2)^2 u^8\,du$	Let $u = \sin(x)$ and $du = \cos(x)\,dx$.
$= \int (u^8 - 2u^{10} + u^{12})\,du$	Expand.
$= \frac{1}{9}u^9 - \frac{2}{11}u^{11} + \frac{1}{13}u^{13} + C$	Evaluate the integral.
$= \frac{1}{9}\sin^9(x) - \frac{2}{11}\sin^{11}(x) + \frac{1}{13}\sin^{13}(x) + C.$	Substitute $u = \sin(x)$.

Evaluating $\int \cos^j(x)\sin^k(x)\,dx$ When k and j are Even

Evaluate $A = \displaystyle\int \sin^4(x)\,dx$.

Solution: Since both the powers of $\sin(x)$ and $\cos(x)$ are even ($k = 4$, $j = 0$), we must use strategy 3. Thus,

$$A = \int (\sin^2(x))^2\,dx \qquad \text{Rewrite } \sin^4(x) = (\sin^2(x))^2.$$

$$= \int \left(\frac{1}{2} - \frac{1}{2}\cos(2x)\right)^2 dx \qquad \text{Substitute } \sin^2(x) = \frac{1}{2} - \frac{1}{2}\cos(2x).$$

$$= \int \left(\frac{1}{4} - \frac{1}{2}\cos(2x) + \frac{1}{4}\cos^2(2x)\right) dx \qquad \text{Expand } \left(\frac{1}{2} - \frac{1}{2}\cos(2x)\right)^2.$$

Since $\cos^2(2x)$ has an even power, we use strategy 3 again and $\cos^2(2x) = \frac{1}{2} + \frac{1}{2}\cos(4x)$

$$= \int \left(\frac{1}{4} - \frac{1}{2}\cos(2x) + \frac{1}{4}\left(\frac{1}{2} + \frac{1}{2}\cos(4x)\right)\right) dx$$

$$= \int \left(\frac{3}{8} - \frac{1}{2}\cos(2x) + \frac{1}{8}\cos(4x)\right) dx \qquad \text{Simplify.}$$

$$= \frac{3}{8}x - \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C. \qquad \text{Evaluate the integral.}$$

Problem Statement

Evaluate $\displaystyle\int \cos^3(x)\,dx$.

Hint: Use strategy 2. Write $\cos^3(x) = \cos^2(x)\cos(x)$ and substitute $\cos^2(x) = 1 - \sin^2(x)$.

Problem Statement

Evaluate $\int \cos^3(x)\,dx$.

Hint: Use strategy 2. Write $\cos^3(x) = \cos^2(x)\cos(x)$ and substitute $\cos^2(x) = 1 - \sin^2(x)$.

Answer: $\sin(x) - \frac{1}{3}\sin^3(x) + C$

Solution

Problem: Evaluate $\int \cos^2(3x)\,dx$.

Hint: Use strategy 3 and substitute $\cos^2(3x) = \frac{1}{2} + \frac{1}{2}\cos(6x)$.

Solution

Problem: Evaluate $\int \cos^2(3x)\,dx$.

Hint: Use strategy 3 and substitute $\cos^2(3x) = \frac{1}{2} + \frac{1}{2}\cos(6x)$.

Solution:

$$\begin{aligned}\int \cos^2(3x)\,dx &= \int \left(\frac{1}{2} + \frac{1}{2}\cos(6x)\right)\,dx\\ &= \frac{1}{2}\int dx + \frac{1}{2}\int \cos(6x)\,dx\\ &= \frac{1}{2}x + \frac{1}{12}\sin(6x) + C\end{aligned}$$

So, the solution is $\frac{1}{2}x + \frac{1}{12}\sin(6x) + C$.

Integrating Products of Sines and Cosines of Different Angles

To integrate products involving $\sin(ax)$, $\sin(bx)$, $\cos(ax)$, and $\cos(bx)$, use the following identities:

$$\begin{aligned}
\sin(ax)\sin(bx) &= \frac{1}{2}\cos\left((a-b)x\right) - \frac{1}{2}\cos\left((a+b)x\right) \\
\sin(ax)\cos(bx) &= \frac{1}{2}\sin\left((a-b)x\right) + \frac{1}{2}\sin\left((a+b)x\right) \\
\cos(ax)\cos(bx) &= \frac{1}{2}\cos\left((a-b)x\right) + \frac{1}{2}\cos\left((a+b)x\right)
\end{aligned}$$

These identities are helpful when dealing with integrals involving products of trigonometric functions with different angles.

Evaluating $\int \sin(ax)\cos(bx)\,dx$

To evaluate $\int \sin(ax)\cos(bx)\,dx$, we can use the identity:

$$\sin(ax)\cos(bx) = \frac{1}{2}\sin((a-b)x) + \frac{1}{2}\sin((a+b)x)$$

Solution:

$$\begin{aligned}\int \sin(5x)\cos(3x)\,dx &= \int \left(\frac{1}{2}\sin(2x) + \frac{1}{2}\sin(8x)\right)\,dx \\ &= -\frac{1}{4}\cos(2x) - \frac{1}{16}\cos(8x) + C\end{aligned}$$

So, $\int \sin(5x)\cos(3x)\,dx = -\frac{1}{4}\cos(2x) - \frac{1}{16}\cos(8x) + C$.

Evaluating $\int \cos(6x)\cos(5x)\,dx$

To evaluate $\int \cos(6x)\cos(5x)\,dx$, we can use the hint provided:

$$\cos(6x)\cos(5x) = \frac{1}{2}\cos(x) + \frac{1}{2}\cos(11x)$$

Evaluating $\int \cos(6x)\cos(5x)\,dx$

To evaluate $\int \cos(6x)\cos(5x)\,dx$, we can use the hint provided:

$$\cos(6x)\cos(5x) = \frac{1}{2}\cos(x) + \frac{1}{2}\cos(11x)$$

Solution:

$$\begin{aligned}\int \cos(6x)\cos(5x)\,dx &= \int \left(\frac{1}{2}\cos(x) + \frac{1}{2}\cos(11x)\right)\,dx \\ &= \frac{1}{2}\sin(x) + \frac{1}{22}\sin(11x) + C\end{aligned}$$

So, $\int \cos(6x)\cos(5x)\,dx = \frac{1}{2}\sin(x) + \frac{1}{22}\sin(11x) + C$.

Integrating Products and Powers of $\tan(x)$ and $\sec(x)$

Before discussing the integration of products of powers of $\tan(x)$ and $\sec(x)$, it is useful to recall the integrals involving $\tan(x)$ and $\sec(x)$ we have already learned:

$$\int \sec^2(x)\,dx = \tan(x) + C,$$

$$\int \sec(x)\tan(x)\,dx = \sec(x) + C,$$

$$\int \tan(x)\,dx = \ln|\sec(x)| + C,$$

$$\int \sec(x)\,dx = \ln|\sec(x) + \tan(x)| + C.$$

Evaluating $\int \sec^j(x)\tan(x)\,dx$

To evaluate $\int \sec^j(x)\tan(x)\,dx$, we can rewrite $\sec^5(x)\tan(x)$ as $\sec^4(x)\sec(x)\tan(x)$. If we let $u = \sec(x)$, then $du = \sec(x)\tan(x)\,dx$, and so

$$\begin{aligned}\int \sec^5(x)\tan(x)\,dx &= \int \sec^4(x)\sec(x)\tan(x)\,dx\\ &= \int u^4\,du\\ &= \frac{1}{5}u^5 + C\\ &= \frac{1}{5}\sec^5(x) + C.\end{aligned}$$

So, $\int \sec^5(x)\tan(x)\,dx = \frac{1}{5}\sec^5(x) + C$.

Evaluating $\int \tan^5(x)\sec^2(x)\,dx$

To evaluate $\int \tan^5(x)\sec^2(x)\,dx$, we can use the hint provided:
Let $u = \tan(x)$ and $du = \sec^2(x)\,dx$.
Solution:

$$\begin{aligned}\int \tan^5(x)\sec^2(x)\,dx &= \int u^5\,du \\ &= \frac{1}{6}u^6 + C \\ &= \frac{1}{6}\tan^6(x) + C\end{aligned}$$

So, $\int \tan^5(x)\sec^2(x)\,dx = \frac{1}{6}\tan^6(x) + C$.

Problem-Solving Strategy: Evaluating $\int \tan^k(x)\sec^j(x)\,dx$

To evaluate $\int \tan^k(x)\sec^j(x)\,dx$, use the following strategies:

- If j is even and $j \geq 2$, rewrite $\sec^j(x) = \sec^{j-2}(x)\sec^2(x)$ and use $\sec^2(x) = \tan^2(x) + 1$ to rewrite $\sec^{j-2}(x)$ in terms of $\tan(x)$. Let $u = \tan(x)$ and $du = \sec^2(x)$.
- If k is odd and $j \geq 1$, rewrite $\tan^k(x)\sec^j(x) = \tan^{k-1}(x)\sec^{j-1}(x)\sec(x)\tan(x)$ and use $\tan^2(x) = \sec^2(x) - 1$ to express $\tan^{k-1}(x)$ in terms of $\sec(x)$. Let $u = \sec(x)$ and $du = \sec(x)\tan(x)\,dx$.
- If k is even and j is odd, then use $\tan^2(x) = \sec^2(x) - 1$ to express $\tan^k(x)$ in terms of $\sec(x)$. Use integration by parts to integrate odd powers of $\sec(x)$.

Evaluating $\int \tan^6(x)\sec^4(x)\,dx$ When j is Even

Since the power on $\sec(x)$ is even, rewrite $\sec^4(x) = \sec^2(x)\sec^2(x)$ and use $\sec^2(x) = \tan^2(x) + 1$ to express the first $\sec^2(x)$ in terms of $\tan(x)$. We now make a substitution $u = \tan(x)$, in which case $du = \sec^2(x)\,dx$, and we obtain

$$\begin{aligned}
\int \tan^6(x)\sec^4(x)\,dx &= \int \tan^6(x)(\tan^2(x)+1)\sec^2(x)dx \\
&= \int u^6(u^2+1)du \\
&= \int (u^8+u^6)du \\
&= \frac{1}{9}u^9 + \frac{1}{7}u^7 + C \\
&= \frac{1}{9}\tan^9(x) + \frac{1}{7}\tan^7(x) + C.
\end{aligned}$$

So, $\int \tan^6(x)\sec^4(x)\,dx = \frac{1}{9}\tan^9(x) + \frac{1}{7}\tan^7(x) + C.$

Evaluating $\int \tan^5(x)\sec^3(x)\,dx$ When k is Odd

Since the power of $\tan(x)$ is odd, we begin by rewriting $\tan^5(x)\sec^3(x) = \tan^4(x)\sec^2(x)\sec(x)\tan(x)$. We then notice that $\tan^4(x) = (\tan^2(x))^2 = (\sec^2(x) - 1)^2$, and make a substitution $u = \sec(x)$ with $du = \sec(x)\tan(x)\,dx$. With this, we obtain

$$\begin{aligned}
\int \tan^5(x)\sec^3(x)\,dx &= \int (\sec^2(x) - 1)^2 \sec^2(x)\sec(x)\tan(x)\,dx \\
&= \int (u^2 - 1)^2 u^2\,du \\
&= \int (u^6 - 2u^4 + u^2)\,du \\
&= \frac{1}{7}u^7 - \frac{2}{5}u^5 + \frac{1}{3}u^3 + C \\
&= \frac{1}{7}\sec^7(x) - \frac{2}{5}\sec^5(x) + \frac{1}{3}\sec^3(x) + C.
\end{aligned}$$

So, $\int \tan^5(x)\sec^3(x)\,dx = \frac{1}{7}\sec^7(x) - \frac{2}{5}\sec^5(x) + \frac{1}{3}\sec^3(x) + C.$

Evaluating $\int \tan^3(x)\,dx$

Although there is no $\sec(x)$ under the integral, we can still use the strategy outlined above for the case when the power k of $\tan(x)$ is odd. For this, we will need to multiply and divide the integrand by $\sec(x)$:

$$\begin{aligned}\tan^3(x) &= \frac{\sec(x)\tan^3(x)}{\sec(x)} = \frac{1}{\sec(x)}\tan^3(x)\sec(x) \\ &= \frac{1}{\sec(x)}\tan^2(x)\sec(x)\tan(x) = \frac{\sec^2(x)-1}{\sec(x)}\sec(x)\tan(x).\end{aligned}$$

Hence, using the substitution $u = \sec(x)$, we obtain

$$\begin{aligned}\int \tan^3(x)\,dx &= \int \frac{\sec^2(x)-1}{\sec(x)}\sec(x)\tan(x)\,dx \\ &= \int \frac{u^2-1}{u}\,du = \int \left(u - \frac{1}{u}\right)\,du \\ &= \frac{1}{2}u^2 - \ln|u| + C = \frac{1}{2}\sec^2(x) - \ln|\sec(x)| + C.\end{aligned}$$

Evaluating $\int \sec^3(x)\,dx$

Integrate $\int \sec^3(x)\,dx$**. Solution:** This integral requires integration by parts. Let $u = \sec(x)$ and $dv = \sec^2(x)\,dx$. These choices make $du = \sec(x)\tan(x)\,dx$ and $v = \tan(x)$. Thus,

$$\begin{aligned}
\int \sec^3(x)\,dx &= \sec(x)\tan(x) - \int \tan(x)\sec(x)\tan(x)\,dx \\
&= \sec(x)\tan(x) - \int \tan^2(x)\sec(x)\,dx \quad \text{(Simplify)} \\
&= \sec(x)\tan(x) - \int (\sec^2(x) - 1)\sec(x)\,dx \quad \text{(Substitute } \tan^2(x) = \sec^2(x) - \\
&= \sec(x)\tan(x) + \int \sec(x)\,dx - \int \sec^3(x)\,dx \quad \text{(Rewrite)} \\
&= \sec(x)\tan(x) + \ln|\sec(x) + \tan(x)| - \int \sec^3(x)\,dx \quad \text{(Evaluate } \int \sec(x)\,d
\end{aligned}$$

Evaluating $\int \sec^3(x)\,dx$ (continued)

We now have

$$\int \sec^3(x)\,dx = \sec(x)\tan(x) + \ln|\sec(x)+\tan(x)| - \int \sec^3(x)\,dx.$$

We see that the last integral is the same as the original one. Let I be a particular antiderivative of $\sec^3(x)$. Substituting I instead of $\int \sec^3(x)\,dx$ into the above equality:

$$I = \sec(x)\tan(x) + \ln|\sec(x)+\tan(x)| - I.$$

Adding I to both sides, we obtain

$$2I = \sec(x)\tan(x) + \ln|\sec(x)+\tan(x)|.$$

Dividing by 2, we arrive at

$$I = \frac{1}{2}\sec(x)\tan(x) + \frac{1}{2}\ln|\sec(x)+\tan(x)|.$$

we obtain that $\int \sec^3(x)\,dx = I + C = \frac{1}{2}\sec(x)\tan(x) + \frac{1}{2}\ln|\sec(x)+\tan(x)| + C.$

Reduction Formulas for $\int \sec^n(x)\,dx$ and $\int \tan^n(x)\,dx$

$$\int \sec^n(x)\,dx = \frac{1}{n-1}\sec^{n-2}(x)\tan(x) + \frac{n-2}{n-1}\int \sec^{n-2}(x)\,dx$$

$$\int \tan^n(x)\,dx = \frac{1}{n-1}\tan^{n-1}(x) - \int \tan^{n-2}(x)\,dx$$

Revisiting $\int \sec^3(x)\,dx$

Apply a reduction formula to evaluate $\int \sec^3(x)\,dx$.

Revisiting $\int \sec^3(x)\,dx$

Apply a reduction formula to evaluate $\int \sec^3(x)\,dx$.

Solution:

By applying the first reduction formula with $n = 3$, we obtain

$$\begin{aligned}&\int \sec^3(x)\,dx\\ &= \frac{1}{3-1}\sec^{3-2}(x)\tan(x) + \frac{3-2}{3-1}\int \sec^{3-2}(x)\,dx\\ &= \frac{1}{2}\sec(x)\tan(x) + \frac{1}{2}\ln|\sec(x)+\tan(x)| + C.\end{aligned}$$

Using a Reduction Formula

Evaluate $\int \tan^4(x)\,dx$.

Using a Reduction Formula

Evaluate $\int \tan^4(x)\,dx$. **Solution:**
Applying the second reduction formula with $n = 4$, we obtain

$$\int \tan^4(x)\,dx = \frac{1}{4-1}\tan^{4-1}(x) - \int \tan^{4-2}(x)\,dx.$$

To evaluate $\int \tan^2(x)\,dx$, we apply the second reduction formula with $n = 2$, which allows us to continue the chain of equalities as follows:

$$\begin{aligned}\int \tan^4(x)\,dx &= \frac{1}{3}\tan^3(x) - \int \tan^2(x)\,dx \\ &= \frac{1}{3}\tan^3(x) - \left(\frac{1}{2-1}\tan^{2-1}(x) - \int \tan^{2-2}(x)\,dx\right) \\ &= \frac{1}{3}\tan^3(x) - \tan(x) + \int 1\,dx = \frac{1}{3}\tan^3(x) - \tan(x) + x + C.\end{aligned}$$

Applying the Reduction Formula

Apply the reduction formula to $\int \sec^5(x)\, dx$.

Applying the Reduction Formula

Apply the reduction formula to $\int \sec^5(x)\,dx$.

Answer:

$$\int \sec^5(x)\,dx = \frac{1}{4}\sec^3(x)\tan(x) - \frac{3}{4}\int \sec^3(x)\,dx$$

Key Concepts

Integrals of trigonometric functions can be evaluated using various strategies. These strategies include the following:

1. Applying trigonometric identities to rewrite the integrand so that it may be evaluated via an appropriate substitution.
2. Using integration by parts.
3. Applying trigonometric identities to rewrite products of sines and cosines with different arguments as the sum of individual sine and cosine functions.
4. Applying reduction formulas.

Understanding and mastering these techniques enables one to effectively evaluate integrals involving trigonometric functions and solve a wide range of mathematical problems.

Key Equations

Sine Products

$$\sin(ax)\sin(bx) = \frac{1}{2}\cos((a-b)x) - \frac{1}{2}\cos((a+b)x)$$

Sine and Cosine Products

$$\sin(ax)\cos(bx) = \frac{1}{2}\sin((a-b)x) + \frac{1}{2}\sin((a+b)x)$$

Cosine Products

$$\cos(ax)\cos(bx) = \frac{1}{2}\cos((a-b)x) + \frac{1}{2}\cos((a+b)x)$$

Power Reduction Formula for Secant

$$\int \sec^n(x)\,dx = \frac{1}{n-1}\sec^{n-1}(x) + \frac{n-2}{n-1}\int \sec^{n-2}(x)\,dx$$

Power Reduction Formula for Tangent

$$\int \tan^n(x)\,dx = \frac{1}{n-1}\tan^{n-1}(x) - \int \tan^{n-2}(x)\,dx$$

3.3 Trigonometric Substitution

Math 1700

University of Manitoba

Winter 2024

Outline

Learning Objectives

Solve integration problems involving the square root of a sum or difference of two squares.

Problem-Solving Strategy: Integrating Expressions Involving $\sqrt{a^2 - x^2}$

- It is a good idea to make sure the integral cannot be evaluated easily in another way. For example, although this method can be applied to integrals of the form $\int \frac{x}{\sqrt{a^2-x^2}}\,dx$ and $\int x\sqrt{a^2 - x^2}\,dx$, they can each be integrated directly by a simple substitution.
- Make the substitution $x = a\sin(\theta)$ and $dx = a\cos(\theta)d\theta$.
- Note: This substitution yields $\sqrt{a^2 - x^2} = a\cos(\theta)$.
- Simplify the expression.
- Evaluate the integral using techniques from the section on trigonometric integrals.
- You may also need to use some trigonometric identities and the relationship $\theta = \sin^{-1}\left(\frac{x}{a}\right)$.

Integrating an Expression Involving $\sqrt{a^2 - x^2}$

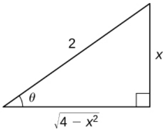
Evaluate $\int \sqrt{4 - x^2}\, dx$.

Solution:

Begin by making the substitutions $x = 2\sin(\theta)$ and $dx = 2\cos(\theta)d\theta$. Since $\sin(\theta) = \frac{x}{2}$, we can construct the reference triangle shown in the following figure.

$\int \sqrt{4-x^2}\,dx = \int \sqrt{4-(2\sin(\theta))^2}\,2\cos(\theta)d\theta$	Substitute $x = 2\sin(\theta)$ and $dx = 2\cos(\theta)d\theta$.
$= \int \sqrt{4\left(1-\sin^2(\theta)\right)}\,2\cos(\theta)d\theta$	Simplify.
$= \int \sqrt{4\cos^2(\theta)}2\cos(\theta)d\theta$	Use the identity $\cos^2(\theta) = 1 - \sin^2(\theta)$.
$= \int 2\lvert\cos(\theta)\rvert 2\cos(\theta)d\theta$	Take the square root.
$= \int 4\cos^2(\theta)d\theta$	Simplify. Since $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, $\cos(\theta) \ge 0$ and $\lvert\cos(\theta)\rvert = \cos(\theta)$.
$= \int 4\left(\frac{1}{2} + \frac{1}{2}\cos(2\theta)\right)d\theta$	Use the strategy for integrating an even power of $\cos(\theta)$.
$= 2\theta + \sin(2\theta) + C$	Evaluate the integral.
$= 2\theta + (2\sin(\theta)\cos(\theta)) + C$	Substitute $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$.
$= 2\sin^{-1}\left(\frac{x}{2}\right) + 2\cdot\frac{x}{2}\cdot\frac{\sqrt{4-x^2}}{2} + C$	Substitute $\sin^{-1}\left(\frac{x}{2}\right) = \theta$ and $\sin(\theta) = \frac{x}{2}$. Use the reference triangle to see that $\cos(\theta) = \frac{\sqrt{4-x^2}}{2}$ and make this substitution.
$= 2\sin^{-1}\left(\frac{x}{2}\right) + \frac{x\sqrt{4-x^2}}{2} + C.$	Simplify.

$\sin\theta = \frac{x}{2}$

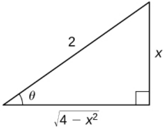


Integrating an Expression Involving $\sqrt{a^2 - x^2}$

Evaluate $\int \frac{\sqrt{4 - x^2}}{x}\, dx$.

Solution: First make the substitutions $x = 2\sin(\theta)$ and $dx = 2\cos(\theta)d\theta$. Since $\sin(\theta) = \frac{x}{2}$, we can construct the reference triangle shown in Figure 3 below.

$\sin\theta = \frac{x}{2}$



$$\int \frac{\sqrt{4-x^2}}{x}\,dx = \int \frac{\sqrt{4-(2\sin(\theta))^2}}{2\sin(\theta)} 2\cos(\theta)d\theta$$

Substitute $x = 2\sin(\theta)$ and $dx = 2\cos(\theta)d\theta$.

$$= \int \frac{2\cos^2(\theta)}{\sin(\theta)} d\theta$$

Substitute $1 - \sin^2(\theta) = \cos^2(\theta)$ and simplify.

$$= \int \frac{2\left(1 - \sin^2(\theta)\right)}{\sin(\theta)} d\theta$$

Substitute $\cos^2(\theta) = 1 - \sin^2(\theta)$.

$$= \int \left(2\csc(\theta) - 2\sin(\theta)\right) d\theta$$

Separate the numerator, simplify, and use $\csc(\theta) = \frac{1}{\sin(\theta)}$.

$$= 2\ln|\csc(\theta) - \cot(\theta)| + 2\cos(\theta) + C$$

Evaluate the integral.

$$= 2\ln\left|\frac{2}{x} - \frac{\sqrt{4-x^2}}{x}\right| + \sqrt{4-x^2} + C.$$

Use the reference triangle to rewrite the expression in terms of x and simplify.

Integrating an Expression Involving $\sqrt{a^2-x^2}$ Two Ways

Method 1: Using the substitution $u = 1 - x^2$

Solution: Let $u = 1 - x^2$, hence $x^2 = 1 - u$. Thus, $du = -2x\,dx$. In this case, the integral becomes

$$\begin{aligned}
\int x^3\sqrt{1-x^2}\,dx &= -\frac{1}{2}\int x^2\sqrt{1-x^2}(-2x\,dx) \quad \text{(Make the substitution)} \\
&= -\frac{1}{2}\int (1-u)\sqrt{u}\,du \quad \text{(Expand the expression)} \\
&= -\frac{1}{2}\int (u^{1/2}-u^{3/2})\,du \quad \text{(Evaluate the integral)} \\
&= -\frac{1}{2}\left(\frac{2}{3}u^{3/2}-\frac{2}{5}u^{5/2}\right)+C \quad \text{(Rewrite in terms of } x\text{)} \\
&= -\frac{1}{3}(1-x^2)^{3/2}+\frac{1}{5}(1-x^2)^{5/2}+C.
\end{aligned}$$

Integrating an Expression Involving $\sqrt{a^2 - x^2}$ Two Ways

Method 2: Using trigonometric substitution $x = \sin(\theta)$

Solution: Let $x = \sin(\theta)$. In this case, $dx = \cos(\theta)d\theta$. Using this substitution, we have

$$\begin{aligned}
\int x^3\sqrt{1-x^2}\,dx &= \int \sin^3(\theta)\cos^2(\theta)\,d\theta \\
&= \int (1-\cos^2(\theta))\cos^2(\theta)\sin(\theta)\,d\theta \quad (\text{Let } u = \cos(\theta)) \\
&= \int (u^4 - u^2)\,du \\
&= \frac{1}{5}u^5 - \frac{1}{3}u^3 + C \quad (\text{Substitute } u = \cos(\theta)) \\
&= \frac{1}{5}\cos^5(\theta) - \frac{1}{3}\cos^3(\theta) + C \quad (\text{Use a reference triangle to} \\
&= \frac{1}{5}(1-x^2)^{5/2} - \frac{1}{3}(1-x^2)^{3/2} + C.
\end{aligned}$$

Integrating an Expression

Using Trigonometric Substitution

Rewrite the integral:

$$\int \frac{x^3}{\sqrt{25 - x^2}}\, dx$$

Answer:

$$\int 125 \sin^3(\theta)\, d\theta$$

Hint: Substitute $x = 5\sin(\theta)$ and $dx = 5\cos(\theta) d\theta$.

Problem-Solving Strategy: Integrating Expressions Involving $\sqrt{a^2 + x^2}$

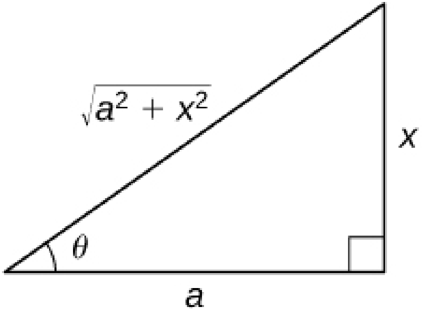
Check to see whether the integral can be evaluated easily by using another method. In some cases, it is more convenient to use an alternative method. **Substitute** $x = a\tan(\theta)$ and $dx = a\sec^2(\theta)\,d\theta$. This substitution yields:

$$\sqrt{a^2 + x^2} = \sqrt{a^2 + (a\tan(\theta))^2} = \sqrt{a^2\left(1 + \tan^2(\theta)\right)}$$
$$= \sqrt{a^2\sec^2(\theta)} = |a\sec(\theta)| = a\sec(\theta).$$

(Since $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ and $\sec(\theta) > 0$ over this interval, $|a\sec(\theta)| = a\sec(\theta)$.)

- Simplify the expression.
- Evaluate the integral using techniques from the section on trigonometric integrals.
- Use the reference triangle from to rewrite the result in terms of x. You may also need to use some trigonometric identities and the relationship $\theta = \tan^{-1}\left(\frac{x}{a}\right)$.
- (Note: The reference triangle is based on the assumption that $x > 0$; however, the trigonometric ratios produced from the reference triangle are the same as the ratios for which $x \leq 0$.)

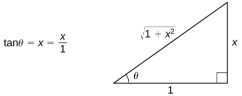
$$\tan\theta = \frac{x}{a}$$



Integrating an Expression Involving $\sqrt{a^2 + x^2}$

Evaluate $\int \frac{dx}{\sqrt{1+x^2}}$

Solution: Begin with the substitution $x = \tan(\theta)$ and $dx = \sec^2(\theta)\, d\theta$. Since $\tan(\theta) = x$, draw the reference triangle in the following figure.



This figure is a right triangle. It has an angle labeled θ. This angle is opposite the vertical side. The hypotenuse is labeled $\sqrt{1+x^2}$, the vertical leg is labeled x, and the horizontal leg is labeled 1. To the left of the triangle is the equation $\tan(\theta) = x/1$. Thus,

$$\int \frac{dx}{\sqrt{1+x^2}} = \int \frac{\sec^2(\theta)}{\sec(\theta)} d\theta \quad \text{(Substitute } x = \tan(\theta) \text{ and } dx = \sec^2(\theta)d\theta\text{. This substitutio}$$

$$= \int \sec(\theta) d\theta \quad \text{(Evaluate the integral.)}$$

$$= \ln|\sec(\theta) + \tan(\theta)| + C \quad \text{(Use the reference triangle to express the result in}$$

$$= \ln|\sqrt{1+x^2} + x| + C.$$

Checking the Solution by Differentiation

To check the solution, differentiate:

$$\begin{aligned}\frac{d}{dx}\left(\ln\left|\sqrt{1+x^2}+x\right|\right) &= \frac{1}{\sqrt{1+x^2}+x}\cdot\left(\frac{x}{\sqrt{1+x^2}}+1\right)\\ &= \frac{1}{\sqrt{1+x^2}+x}\cdot\frac{x+\sqrt{1+x^2}}{\sqrt{1+x^2}}\\ &= \frac{1}{\sqrt{1+x^2}}.\end{aligned}$$

Since $\sqrt{1+x^2}+x>0$ for all values of x, we could rewrite $\ln\left|\sqrt{1+x^2}+x\right|+C$ as $\ln(\sqrt{1+x^2}+x)+C$, if desired.

Evaluating $\int \frac{dx}{\sqrt{1+x^2}}$ Using a Different Substitution

Using $x = \sinh(\theta)$

Solution: Because $\sinh(\theta)$ has a range of all real numbers, and $1 + \sinh^2(\theta) = \cosh^2(\theta)$, we may also use the substitution $x = \sinh(\theta)$ to evaluate this integral. In this case, $dx = \cosh(\theta)d\theta$. Consequently,

$$\begin{aligned}
\int \frac{dx}{\sqrt{1+x^2}} &= \int \frac{\cosh(\theta)}{\sqrt{1+\sinh^2(\theta)}} d\theta \quad \text{(Substitute } x = \sinh(\theta) \text{ and } dx = \cosh(\theta)d\theta\text{. This su} \\
&= \int \frac{\cosh(\theta)}{\sqrt{\cosh^2(\theta)}} d\theta = \int \frac{\cosh(\theta)}{|\cosh(\theta)|} d\theta \quad \left(\sqrt{\cosh^2(\theta)} = |\cosh(\theta)|\right) \\
&= \int \frac{\cosh(\theta)}{\cosh(\theta)} d\theta \quad (|\cosh(\theta)| = \cosh(\theta) \text{ since } \cosh(\theta) > 0 \text{ for all } \theta) \\
&= \int 1 \, d\theta \quad \text{(Simplify.)} \\
&= \theta + C \quad \text{(Evaluate the integral. Since } x = \sinh(\theta)\text{, we know } \theta = \sinh^{-1}(x)\text{.)} \\
&= \sinh^{-1}(x) + C.
\end{aligned}$$

Analysis: Comparison of Solutions

This answer looks quite different from the answer obtained using the substitution $x = \tan(\theta)$. To see that the solutions are the same, set $y = \sinh^{-1}(x)$. Then $\sinh y = x$, that is,

$$\frac{e^y - e^{-y}}{2} = x.$$

After multiplying both sides by $2e^y$ and rewriting, this equation becomes:

$$e^{2y} - 2xe^y - 1 = 0.$$

Use the quadratic equation formula to solve for e^y:

$$e^y = \frac{2x \pm \sqrt{4x^2 + 4}}{2}.$$

Simplifying, we have:

$$e^y = x \pm \sqrt{x^2 + 1}.$$

Since $x - \sqrt{x^2 + 1} < 0$, it must be the case that $e^y = x + \sqrt{x^2 + 1}$. Therefore,

$$y = \ln\left(x + \sqrt{x^2 + 1}\right).$$

At last, we obtain:

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right).$$

Analysis: Comparison of Solutions (Continued)

After we make the final observation that, since $x + \sqrt{x^2 + 1} > 0$,

$$\ln\left(x + \sqrt{x^2 + 1}\right) = \ln\left|\sqrt{1 + x^2} + x\right|,$$

we see that the two different methods produced the same solutions.

Conclusion: The solutions obtained using the substitutions $x = \tan(\theta)$ and $x = \sinh(\theta)$ are equivalent. Although they may appear different at first glance, they lead to the same result after careful analysis and simplification.

Finding an Arc Length

Problem: Find the length of the curve $y = x^2$ over the interval $\left[0, \frac{1}{2}\right]$.

Solution: Because $\frac{dy}{dx} = 2x$, the arc length is given by

$$\textbf{Arc Length} = \int_a^b \sqrt{1 + [f'(x)]^2} dx = \int_0^{\frac{1}{2}} \sqrt{1 + (2x)^2}\, dx = \int_0^{\frac{1}{2}} \sqrt{1 + 4x^2}\, dx.$$

To evaluate this integral, use the substitution $x = \frac{1}{2}\tan(\theta)$ and $dx = \frac{1}{2}\sec^2(\theta)d\theta$. We also need to change the limits of integration. If $x = 0$, then $\theta = 0$ and if $x = \frac{1}{2}$, then $\theta = \frac{\pi}{4}$. Thus,

$$\begin{aligned}
\int_0^{\frac{1}{2}} \sqrt{1 + 4x^2}\, dx &= \int_0^{\frac{\pi}{4}} \sqrt{1 + \tan^2(\theta)} \cdot \frac{1}{2}\sec^2(\theta)\, d\theta = \frac{1}{2}\int_0^{\frac{\pi}{4}} \sec^3(\theta)\, d\theta \\
&= \frac{1}{2}\left(\frac{1}{2}\sec(\theta)\tan(\theta) + \frac{1}{2}\ln|\sec(\theta) + \tan(\theta)|\right)\Bigg|_0^{\frac{\pi}{4}} \\
&= \frac{1}{4}\left(\sqrt{2} + \ln\left(\sqrt{2} + 1\right)\right).
\end{aligned}$$

Rewriting the Integral

Problem: Rewrite $\int x^3 \sqrt{x^2 + 4}\, dx$ by using a substitution involving $\tan(\theta)$.

Rewriting the Integral

Problem: Rewrite $\int x^3\sqrt{x^2+4}\,dx$ by using a substitution involving $\tan(\theta)$. **Answer:** We use the substitution $x = 2\tan(\theta)$ and $dx = 2\sec^2(\theta)d\theta$. Thus,

$$\int x^3\sqrt{x^2+4}\,dx = \int (2\tan(\theta))^3\sqrt{(2\tan(\theta))^2+4}\cdot 2\sec^2(\theta)\,d\theta$$
$$= 32\int \tan^3(\theta)\sec^3(\theta)\,d\theta.$$

Hint: Use $x = 2\tan(\theta)$ and $dx = 2\sec^2(\theta)d\theta$.

Problem-Solving Strategy: Integrals Involving $\sqrt{x^2 - a^2}$

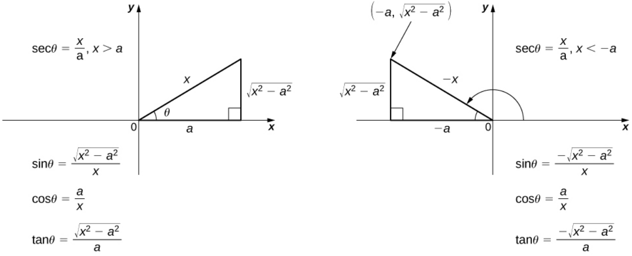
Step 1: Check to see whether the integral cannot be evaluated using another method. If so, we may wish to consider applying an alternative technique.
Step 2: Substitute $x = a\sec(\theta)$ and $dx = a\sec(\theta)\tan(\theta)d\theta$. This substitution yields

$$\sqrt{x^2 - a^2} = \sqrt{(a\sec(\theta))^2 - a^2} = \sqrt{a^2\left(\sec^2(\theta) - 1\right)} = \sqrt{a^2\tan^2(\theta)}$$
$$= a|\tan(\theta)|.$$

For $x \geq a$, we have $\theta \in \left[0, \frac{\pi}{2}\right)$, which implies that $\tan(\theta) \geq 0$, and so $a|\tan(\theta)| = a\tan(\theta)$ while for $x \leq -a$, $\theta \in \left(\frac{\pi}{2}, \pi\right]$, implying that $\tan(\theta) \leq 0$, and hence $a|\tan(\theta)| = -a\tan(\theta)$.
Step 3: Simplify the expression.
Step 4: Evaluate the integral using techniques from the section on trigonometric integrals.
Step 5: Use the reference triangles to rewrite the result in terms of x. You may also need to use some trigonometric identities and the relationship $\theta = \sec^{-1}\left(\frac{x}{a}\right)$.
Note: We need both reference triangles, since the values of some of the trigonometric ratios are different depending on whether $x \geq a$ or $x \leq -a$.

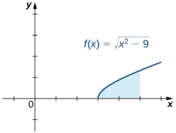
Problem-Solving Strategy: Integrals Involving $\sqrt{x^2 - a^2}$ (continued)

Note (continued): There are also the equations $\sin(\theta) = \frac{\sqrt{x^2 - a^2}}{x}$, $\cos(\theta) = \frac{a}{x}$, and $\tan(\theta) = \frac{\sqrt{x^2 - a^2}}{a}$. The second triangle is in the second quadrant, with the hypotenuse labeled $-x$. The horizontal leg is labeled $-a$ and is on the negative x-axis. The vertical leg is labeled $\sqrt{x^2 - a^2}$. To the right of the triangle is the equation $\sec(\theta) = \frac{x}{a}$.



Finding the Area of a Region

Problem: Find the area of the region between the graph of $f(x) = \sqrt{x^2 - 9}$ and the x-axis over the interval $[3, 5]$.
Solution: First, sketch a rough graph of the region described in the problem.



We can see that the area is $A = \int_3^5 \sqrt{x^2 - 9}\, dx$. To evaluate this definite integral, substitute $x = 3\sec(\theta)$ and $dx = 3\sec(\theta)\tan(\theta)d\theta$. We must also change the limits of integration. If $x = 3$, then $3 = 3\sec(\theta)$ and hence $\theta = 0$. If $x = 5$, then $\theta = \sec^{-1}\left(\frac{5}{3}\right)$.

Finding the Area of a Region (continued)

After making these substitutions and simplifying, we have:

$$\begin{aligned}
\text{Area} &= \int_3^5 \sqrt{x^2 - 9}\, dx = \int_0^{\sec^{-1}\left(\frac{5}{3}\right)} 9\tan^2(\theta)\sec(\theta)d\theta \quad (\text{since } \tan^2(\theta) = 1 - \sec^2(\theta)) \\
&= \int_0^{\sec^{-1}\left(\frac{5}{3}\right)} 9\left(\sec^2(\theta) - 1\right)\sec(\theta)d\theta \quad (\text{expand}) \\
&= \int_0^{\sec^{-1}\left(\frac{5}{3}\right)} 9\left(\sec^3(\theta) - \sec(\theta)\right) d\theta \quad (\text{evaluate the integral}) \\
&= \left(\frac{9}{2}\ln|\sec(\theta) + \tan(\theta)| + \frac{9}{2}\sec(\theta)\tan(\theta)\right) - 9\ln|\sec(\theta) + \tan(\theta)|\Big|_0^{\sec^{-1}\left(\frac{5}{3}\right)} \quad (\text{simplify} \\
&= \frac{9}{2}\sec(\theta)\tan(\theta) - \frac{9}{2}\ln|\sec(\theta) + \tan(\theta)| + \tan(\theta)|\Big|_0^{\sec^{-1}\left(\frac{5}{3}\right)} \quad (\text{evaluate}) \\
&= 10 - \frac{9}{2}\ln 3.
\end{aligned}$$

Solution (continued): The final area of the region between the graph of $f(x) = \sqrt{x^2 - 9}$ and the x-axis over the interval $[3, 5]$ is $10 - \frac{9}{2}\ln 3$.

Evaluating $\int \frac{dx}{\sqrt{x^2-4}}$

Problem: Evaluate $\int \frac{dx}{\sqrt{x^2-4}}$. Assume that $x > 2$.

Answer: $\ln\left|\frac{x}{2} + \frac{\sqrt{x^2-4}}{2}\right| + C$

Hint: Substitute $x = 2\sec(\theta)$ and $dx = 2\sec(\theta)\tan(\theta)d\theta$.

Key Concepts

- For integrals involving $\sqrt{a^2 - x^2}$, use the substitution $x = a\sin(\theta)$ and $dx = a\cos(\theta)d\theta$.
- For integrals involving $\sqrt{a^2 + x^2}$, use the substitution $x = a\tan(\theta)$ and $dx = a\sec^2(\theta)d\theta$.
- For integrals involving $\sqrt{x^2 - a^2}$, substitute $x = a\sec(\theta)$ and $dx = a\sec(\theta)\tan(\theta)d\theta$.

3.4 Partial Fractions

Math 1700

University of Manitoba

Winter 2024

Outline

1. Some techniques
2. The General Method
3. Simple Quadratic Factors

Learning Objectives

- Integrate a rational function using the method of partial fractions.
- Recognize simple linear factors in a rational function.
- Recognize repeated linear factors in a rational function.
- Recognize quadratic factors in a rational function.

Informations

We have seen some techniques for integrating specific rational functions:

- Integration of $\frac{du}{u}$ leads to $\ln|u| + C$, which yields:

$$\int \frac{dx}{ax+b} = \frac{1}{a}\ln|ax+b| + C \quad (a \neq 0)$$

- Integration of $\frac{dx}{x^2+a^2}$ using trigonometric substitution results in:

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C \quad (a > 0)$$

Problem!

- However, we still lack a technique for arbitrary polynomial quotients, such as $\int \frac{3x}{x^2-x-2}\,dx$.
- Partial fraction decomposition allows us to decompose such rational functions into simpler forms.
- It's essential to understand the form of decomposition, dependent on the factorization of the denominator.
- Which approach when $\deg(P(x)) > \deg(Q(x))$
- Remember, partial fraction decomposition applies only when $\deg(P(x)) < \deg(Q(x))$. If not, perform long division first.

Integrating $\int \frac{P(x)}{Q(x)}\,dx$, where $\deg(P(x)) \geq \deg(Q(x))$

Evaluate $\int \frac{x^2+3x+5}{x+1}\,dx$.

Integrating $\int \frac{P(x)}{Q(x)}\,dx$, where $\deg(P(x)) \geq \deg(Q(x))$

Evaluate $\int \frac{x^2+3x+5}{x+1}\,dx$. **Solution:** Since $\deg(x^2+3x+5) = 2 > 1 = \deg(x+1)$, we perform long division to obtain

$$\frac{x^2+3x+5}{x+1} = x+2+\frac{3}{x+1}.$$

Thus,

$$\int \frac{x^2+3x+5}{x+1}\,dx = \int \left(x+2+\frac{3}{x+1}\right)\,dx$$
$$= \frac{1}{2}x^2+2x+3\ln|x+1|+C.$$

Problem-Solving Strategy: Partial Fraction Decomposition

- For each irreducible quadratic factor $ax^2 + bx + c$ that $Q(x)$ contains, the decomposition must include

$$\frac{Ax + B}{ax^2 + bx + c}.$$

- For each repeated irreducible quadratic factor $(ax^2 + bx + c)^n$, the decomposition must include

$$\frac{A_1 x + B_1}{ax^2 + bx + c} + \frac{A_2 x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_n x + B_n}{(ax^2 + bx + c)^n}.$$

- After the appropriate decomposition is determined, solve for the constants.
- If using the decomposition to evaluate an integral, rewrite the integrand in its decomposed form and evaluate it using previously developed techniques or integration formulas.
- If using the decomposition to evaluate an integral, rewrite the integrand in its decomposed form and evaluate it using previously developed techniques or integration formulas.

Evaluate $\int \frac{x-3}{x+2}\,dx$

Answer:

$$\int \frac{x-3}{x+2}\,dx = x - 5\ln|x+2| + C$$

Hint: Use long division to obtain $\frac{x-3}{x+2} = 1 - \frac{5}{x+2}$.

Integrating Rational Functions $\deg(P(x)) < \deg(Q(x))$

To integrate $\int \frac{P(x)}{Q(x)}\, dx$, where $\deg(P(x)) < \deg(Q(x))$, we must begin by factoring $Q(x)$.

Nonrepeated Linear Factors:

If $Q(x)$ can be factored as $(a_1x + b_1)(a_2x + b_2)\dots(a_nx + b_n)$, where each linear factor is distinct and no factor is a constant multiple of another, then it is possible to find constants $A_1, A_2, \dots, A_n$ satisfying

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \cdots + \frac{A_n}{a_nx + b_n}.$$

Partial Fractions with Nonrepeated Linear Factors

Evaluate $\int \frac{3x+2}{x^3-x^2-2x}\,dx$.

Partial Fractions with Nonrepeated Linear Factors

Evaluate $\int \frac{3x+2}{x^3-x^2-2x}\,dx$. **Solution:** Since $\deg(3x+2) = 1 < 3 = \deg(x^3 - x^2 - 2x)$, we begin by factoring the denominator of the integrand. We can see that $x^3 - x^2 - 2x = x(x-2)(x+1)$. Thus, there are constants A, B, and C satisfying

$$\frac{3x+2}{x(x-2)(x+1)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+1}.$$

We must now find these constants. To do so, we begin by bringing the right-hand side to a common denominator. We have:

$$\frac{3x+2}{x(x-2)(x+1)} = \frac{A(x-2)(x+1) + Bx(x+1) + Cx(x-2)}{x(x-2)(x+1)}.$$

Now, we set the numerators equal to each other, obtaining

$$3x + 2 = A(x-2)(x+1) + Bx(x+1) + Cx(x-2).$$

Method of Equating Coefficients

Expand the right-hand side of (1) and then group the terms by the powers of x to rewrite it as:

$$3x + 2 = (A + B + C)x^2 + (-A + B - 2C)x + (-2A)$$

Equating coefficients produces the system of equations:

$$\begin{aligned} A + B + C &= 0 \\ -A + B - 2C &= 3 \\ -2A &= 2 \end{aligned}$$

Method of Equating Coefficients (cont'd)

To solve this system, we first observe that $-2A = 2 \to A = -1$. Substituting this value into the first two equations gives us the system:

$$\begin{aligned} B + C &= 1 \\ B - 2C &= 2 \end{aligned}$$

Multiplying the second equation by -1 and adding the resulting equation to the first produces:

$$-3C = 1$$

which in turn implies that $C = -\frac{1}{3}$. Substituting this value into the equation $B + C = 1$ yields $B = \frac{4}{3}$. Thus, solving these equations yields $A = -1$, $B = \frac{4}{3}$, and $C = -\frac{1}{3}$.

Method of Strategic Substitution

The method of strategic substitution is based on the assumption that we have set up the decomposition correctly. If the decomposition is set up correctly, then there must be values of A, B, and C that satisfy (1) for all values of x. That is, this equation must be true for any value of x we care to substitute into it. Therefore, by choosing values of x carefully and substituting them into the equation, we may find A, B, and C easily. For example, If we substitute

- $x = 0$, the equation reduces to $2 = A(-2)(1)$, yields $A = -1$.
- $x = 2$, the equation reduces to $8 = B(2)(3)$, or equivalently $B = \frac{4}{3}$.
- $x = -1$ into the equation and obtain $-1 = C(-1)(-3)$. Then $C = -\frac{1}{3}$.

It is important to keep in mind that if we attempt to use this method with a decomposition that has not been set up correctly, we are still able to find values for the constants, but these constants are meaningless. If we do opt to use the method of strategic substitution, then it is a good idea to check the result by recombining the terms algebraically.

Dividing before Applying Partial Fractions

Evaluate $\int \frac{x^2+3x+1}{x^2-4}\,dx$. **Solution:** Since $\deg(x^2+3x+1) = 2 = \deg(x^2-4)$, we must perform long division of polynomials. This results in:

$$\frac{x^2+3x+1}{x^2-4} = 1 + \frac{3x+5}{x^2-4}.$$

Next, we perform partial fraction decomposition on:

$$\frac{3x+5}{x^2-4} = \frac{3x+5}{(x+2)(x-2)}.$$

We have:

$$\frac{3x+5}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2}.$$

Thus:

$$3x+5 = A(x+2) + B(x-2).$$

Solving for A and B using either method, we obtain $A = \frac{11}{4}$ and $B = \frac{1}{4}$.

Applying Partial Fractions after a Substitution

Evaluate $\int \frac{\cos(x)}{\sin^2(x)-\sin(x)}\,dx$.

Solution: Let's begin by letting $u = \sin(x)$. Consequently, $du = \cos(x)\,dx$. After making these substitutions, we have:

$$\int \frac{\cos(x)}{\sin^2(x) - \sin(x)}\,dx = \int \frac{du}{u^2 - u} = \int \frac{du}{u(u-1)}.$$

Applying partial fraction decomposition to $\frac{1}{u(u-1)}$ gives:

$$\frac{1}{u(u-1)} = -\frac{1}{u} + \frac{1}{u-1}.$$

Therefore:

$$\begin{aligned}\int \frac{\cos(x)}{\sin^2(x) - \sin(x)}\,dx &= -\ln|u| + \ln|u-1| + C \\ &= -\ln|\sin(x)| + \ln|\sin(x) - 1| + C.\end{aligned}$$

Evaluate $\int \frac{x+1}{(x+3)(x-2)}\,dx$.

Answer:

$$\frac{2}{5}\ln|x+3| + \frac{3}{5}\ln|x-2| + C$$

Hint:

$$\frac{x+1}{(x+3)(x-2)} = \frac{A}{x+3} + \frac{B}{x-2}$$

Repeated Linear Factors $\int \frac{P(x)}{(ax+b)^n}$

For some applications, we need to integrate rational expressions that have denominators with repeated linear factors—that is, there is at least one factor of the form $(ax+b)^n$, where n is a positive integer greater than or equal to 2. If the denominator contains the repeated linear factor $(ax+b)^n$, then the corresponding terms in the decomposition are:

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n}.$$

Partial Fractions with Repeated Linear Factors

Evaluate $\int \frac{x-2}{(2x-1)^2(x-1)}\,dx$. **Solution:** We have $\deg(x-2) = 1 < 3 = \deg((2x-1)^2(x-1))$, so we can proceed with the decomposition. Since $(2x-1)^2$ is a repeated linear factor, the corresponding terms in the decomposition are going to be $\frac{A}{2x-1} + \frac{B}{(2x-1)^2}$, and hence

$$\frac{x-2}{(2x-1)^2(x-1)} = \frac{A}{2x-1} + \frac{B}{(2x-1)^2} + \frac{C}{x-1}.$$

Bringing to a common denominator and equating the numerators, we have:

$$x-2 = A(2x-1)(x-1) + B(x-1) + C(2x-1)^2.$$

We then use the method of equating coefficients to find the values of A, B, and C.

$$x-2 = (2A+4C)x^2 + (-3A+B-4C)x + (A-B+C).$$

Equating coefficients yields $2A+4C=0$, $-3A+B-4C=1$, and $A-B+C=-2$. Solving this system we obtain that $A=2$, $B=3$, and $C=-1$.

Partial Fractions with Repeated Linear Factors (Contd.)

Alternatively, we can use the method of strategic substitution. In this case, substituting $x = 1$ and $x = 1/2$ into the equation easily produces the values $B = 3$ and $C = -1$. At this point, it may seem that we have run out of good choices for x, however, since we already have values for B and C, we can substitute in these values and choose any x that we haven't used yet. The value $x = 0$ is a good option since it's very easy to substitute. This way, we obtain:

$$-2 = A(-1)(-1) + 3(-1) + (-1)(-1)^2,$$

and solving for A we get $A = 2$. Now that we have the values for A, B, and C, we rewrite the original integral:

$$\begin{aligned}\int \frac{x-2}{(2x-1)^2(x-1)}\,dx &= \int \left(\frac{2}{2x-1} + \frac{3}{(2x-1)^2} - \frac{1}{x-1}\right)\,dx \\ &= \ln|2x-1| - \frac{3}{2(2x-1)} - \ln|x-1| + C.\end{aligned}$$

To integrate $\frac{3}{(2x-1)^2}$, we make a substitution $u = 2x - 1$, yielding $du = 2dx$, and then use the power formula to evaluate: $\int \frac{3}{(2x-1)^2}\,dx = \frac{3}{2}\int u^{-2}\,du$.

Partial Fraction Decomposition Setup

Set up the partial fraction decomposition for $\frac{x+2}{(x+3)^3(x-4)^2}$. (Do not solve for the coefficients or perform integration.)

Answer:

$$\frac{x+2}{(x+3)^3(x-4)^2} = \frac{A}{x+3} + \frac{B}{(x+3)^2} + \frac{C}{(x+3)^3} + \frac{D}{x-4} + \frac{E}{(x-4)^2}$$

Problem-Solving Strategy: Partial Fraction Decomposition

To decompose the rational function $P(x)/Q(x)$, use the following steps:

1. Make sure that $\deg(P(x)) < \deg(Q(x))$. If not, perform long division of polynomials.
2. Factor $Q(x)$ into the product of linear and irreducible quadratic factors. An irreducible quadratic is a quadratic that has no real zeros.
3. Assuming that $\deg(P(x)) < \deg(Q(x))$, the factors of $Q(x)$ determine the form of the decomposition of $P(x)/Q(x)$.
   - If $Q(x)$ can be factored as $(a_1x + b_1)(a_2x + b_2)...(a_nx + b_n)$, where each linear factor is distinct and no factor is a constant multiple of another, then it is possible to find constants $A_1, A_2, ...A_n$ satisfying
   $$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \cdots + \frac{A_n}{a_nx + b_n}.$$
   - If $Q(x)$ contains the repeated linear factor $(ax + b)^n$, then the decomposition must contain.
   $$\frac{A_1}{ax + b} + \frac{A_2}{(ax + b)^2} + \cdots + \frac{A_n}{(ax + b)^n}.$$

Problem-Solving Strategy: Partial Fraction Decomposition

- For each irreducible quadratic factor $ax^2 + bx + c$ that $Q(x)$ contains, the decomposition must include
$$\frac{Ax + B}{ax^2 + bx + c}.$$
- For each repeated irreducible quadratic factor $(ax^2 + bx + c)^n$, the decomposition must include
$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_nx + B_n}{(ax^2 + bx + c)^n}.$$
- After the appropriate decomposition is determined, solve for the constants.
- If using the decomposition to evaluate an integral, rewrite the integrand in its decomposed form and evaluate it using previously developed techniques or integration formulas.
- If using the decomposition to evaluate an integral, rewrite the integrand in its decomposed form and evaluate it using previously developed techniques or integration formulas.

Rational Expressions with an Irreducible Quadratic Factor

Now let's look at integrating a rational expression in which the denominator contains an irreducible quadratic factor. Recall that the quadratic $ax^2 + bx + c$ is irreducible if $ax^2 + bx + c = 0$ has no real zeros—that is, if $b^2 - 4ac < 0$.

Evaluate $\int \frac{2x-3}{x^3+x} dx$.

Solution

Since $\deg(2x-3) = 1 < 3 = \deg(x^3+x)$, factor the denominator and proceed with partial fraction decomposition. Because $x^3 + x = x(x^2+1)$ contains irreducible quadratic factor x^2+1, include $\frac{Ax+B}{x^2+1}$ as a part of the decomposition, along with $\frac{C}{x}$ for the linear term x. Thus, the decomposition has the form

$$\frac{2x-3}{x(x^2+1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x}.$$

Rational Expressions with an Irreducible Quadratic Factor

After bringing to a common denominator and equating the numerators, we obtain the equation

$$2x - 3 = (Ax + B)x + C(x^2 + 1).$$

Solving for A, B, and C, we get $A = 3$, $B = 2$, and $C = -3$. Therefore,

$$\frac{2x-3}{x^3+x} = \frac{3x+2}{x^2+1} - \frac{3}{x}.$$

Substituting back into the integral, we obtain

$$\begin{aligned}\int \frac{2x-3}{x^3+x}dx &= \int \left(\frac{3x+2}{x^2+1} - \frac{3}{x}\right) dx \\ &= 3\int \frac{x}{x^2+1}dx + 2\int \frac{1}{x^2+1}dx - 3\int \frac{1}{x}dx \\ &= \frac{3}{2}\ln|x^2+1| + 2\tan^{-1} x - 3\ln|x| + C.\end{aligned}$$

In order to evaluate $\int \frac{x}{x^2+1}dx$, we perform a substitution $u = x^2 + 1$. Note: We may rewrite $\ln|x^2+1| = \ln(x^2+1)$, if we wish to do so, since $x^2 + 1 > 0$.

Partial Fractions with an Irreducible Quadratic Factor 1

Evaluate $\int \frac{dx}{x^3-8}$. **Solution** Since the numerator is 1 and $\deg(1) = 0 < 3 = \deg(x^3-8)$, we can proceed with partial fraction decomposition. We start by factoring $x^3 - 8 = (x-2)(x^2+2x+4)$. We see that the quadratic factor x^2+2x+4 is irreducible since $2^2 - 4(1)(4) = -12 < 0$. Using the decomposition described in the problem-solving strategy, we get

$$\frac{1}{(x-2)(x^2+2x+4)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+2x+4}.$$

After bringing to a common denominator and equating the numerators, this becomes

$$1 = A(x^2+2x+4) + (Bx+C)(x-2).$$

Applying either method, we get $A = \frac{1}{12}$, $B = -\frac{1}{12}$, and $C = -\frac{1}{3}$. Rewriting $\int \frac{dx}{x^3-8}$, we have

$$\int \frac{dx}{x^3-8} = \frac{1}{12}\int \frac{1}{x-2}dx - \frac{1}{12}\int \frac{x+4}{x^2+2x+4}dx.$$

Partial Fractions with an Irreducible Quadratic Factor 2

We can see that

$$\int \frac{1}{x-2}\,dx = \ln|x-2| + C,$$

but $\int \frac{x+4}{x^2+2x+4}\,dx$ requires a bit more effort. Let's begin by completing the square in x^2+2x+4 to obtain

$$x^2+2x+4 = (x+1)^2+3.$$

By letting $u = x+1$ and consequently $du = dx$, we see that

$$\begin{aligned}\int \frac{x+4}{x^2+2x+4}\,dx &= \int \frac{u+3}{u^2+3}\,du \\ &= \int \frac{u}{u^2+3}\,du + \int \frac{3}{u^2+3}\,du.\end{aligned}$$

Splitting the numerator apart, we get

$$\int \frac{u}{u^2+3}\,du + \int \frac{3}{u^2+3}\,du = \frac{1}{2}\ln|u^2+3| + \frac{3}{\sqrt{3}}\tan^{-1}\left(\frac{u}{\sqrt{3}}\right) + C.$$

End

Replace $u = x + 1$

$$\frac{1}{2}\ln|x^2 + 2x + 4| + \sqrt{3}\tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right) + C.$$

Substituting back into the original integral and simplifying gives

$$\int \frac{dx}{x^3 - 8} = \frac{1}{12}\ln|x - 2| - \frac{1}{24}\ln|x^2 + 2x + 4| - \frac{\sqrt{3}}{12}\tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right) + C.$$

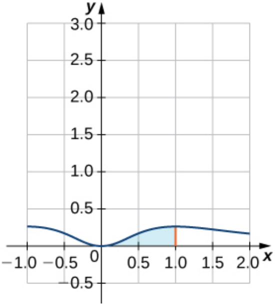
Here again, we can drop the absolute value if we wish to do so, since $x^2 + 2x + 4 > 0$ for all x.

Finding a Volume

Find the volume of the solid of revolution obtained by revolving the region enclosed by the graph of $f(x) = \frac{x^2}{(x^2+1)^2}$ and the x-axis over the interval $[0, 1]$ about the y-axis.

Solution

Let's begin by sketching the region to be revolved. From the sketch, we see that the shell method is a good choice for solving this problem.



Solution (Contd.)

The volume is given by

$$V = 2\pi \int_0^1 x \cdot \frac{x^2}{(x^2+1)^2}\, dx = 2\pi \int_0^1 \frac{x^3}{(x^2+1)^2}\, dx$$

Since $\deg(x^3) = 3 < 4 = \deg((x^2+1)^2)$, we can proceed with partial fraction decomposition.

$$\frac{x^3}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}$$

After finding $A = 1$, $B = 0$, $C = -1$, and $D = 0$, we substitute back into the integral:

$$V = \pi(\ln(2) - \frac{1}{2})$$

Partial Fraction Decomposition Setup

We aim to find the partial fraction decomposition for the expression:

$$\frac{x^2+3x+1}{(x+2)(x-3)^2(x^2+4)^2}$$

We express it as the sum of simpler fractions:

$$\frac{x^2+3x+1}{(x+2)(x-3)^2(x^2+4)^2} = \frac{A}{x+2}+\frac{B}{x-3}+\frac{C}{(x-3)^2}+\frac{Dx+E}{x^2+4}+\frac{Fx+G}{(x^2+4)^2}$$

Now, we need to determine the values of coefficients A, B, C, D, E, F, and G.

Key Concepts

- Partial fraction decomposition is a technique used to break down a rational function into a sum of simple rational functions that can be integrated using previously learned techniques.
- When applying partial fraction decomposition, we must ensure that the degree of the numerator is less than the degree of the denominator. If not, we need to perform long division before attempting partial fraction decomposition.
- The form the decomposition takes depends on the type of factors in the denominator. These types include:
  - Nonrepeated linear factors
  - Repeated linear factors
  - Nonrepeated irreducible quadratic factors
  - Repeated irreducible quadratic factors

3.7 Improper Integrals

Math 1700

University of Manitoba

Winter 2024

Outline

Learning Objectives

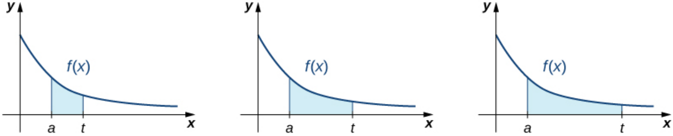
- Evaluate an integral over an infinite interval.
- Evaluate an integral over a closed interval with an infinite discontinuity within the interval.
- Use the comparison theorem to determine whether a definite integral is convergent.

Introduction

To define the integral $\int_a^\infty f(x)\,dx$, we interpret it as the limit of the definite integral $\int_a^t f(x)\,dx$ as t approaches infinity:

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

In the figure below, we visually interpret this definition:



Definition

Let $f(x)$ be continuous over an interval $[a, \infty)$. Then

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx, \text{ provided this limit exists.}$$

Let $f(x)$ be continuous over an interval of the form $(-\infty, b]$. Then

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx, \text{ provided this limit exists.}$$

Convergence

If the limit exists, then the improper integral is said to converge. If the limit does not exist, then the improper integral is said to diverge

Definition

Let $f(x)$ be continuous over $(-\infty, \infty)$. We define

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx,$$

provided that both $\int_{-\infty}^{0} f(x)\,dx$ and $\int_{0}^{\infty} f(x)\,dx$ converge.

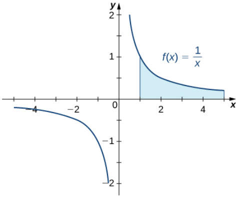
If either of these two integrals is divergent, then $\int_{-\infty}^{\infty} f(x)\,dx$ diverges. (It can be shown that, in fact, $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx$ for any value of a.)

Finding an Area

Determine whether the area between the graph of $f(x) = \frac{1}{x}$ and the x-axis over the interval $[1, \infty)$ is finite or infinite.

Solution:

- We first do a quick sketch of the region in question, as shown in the following graph.



We can find the area between the curve $f(x) = \frac{1}{x}$ and the x-axis on an infinite interval.

- We can see that the area of this region is given by $A = \int_1^\infty \frac{1}{x}\, dx$.

Solution

Then we have

$$\begin{aligned}
A &= \int_1^\infty \frac{1}{x}\,dx \\
&= \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx \quad \text{Rewrite the improper integral as a limit.} \\
&= \lim_{t\to\infty} \ln|x|\Big|_1^t \quad \text{Find the antiderivative.} \\
&= \lim_{t\to\infty} (\ln|t| - \ln(1)) \quad \text{Evaluate the antiderivative.} \\
&= \infty \quad \text{Evaluate the limit.}
\end{aligned}$$

Since the improper integral diverges to ∞, the area of the region is infinite.

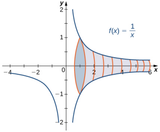
Finding a Volume

Find the volume of the solid obtained by revolving the region bounded by the graph of $f(x) = \frac{1}{x}$ and the x-axis over the interval $[1, \infty)$ about the x-axis.

Solution:

- The solid is shown in Figure below. Using the disk method, we see that the volume V is

$$V = \int_1^\infty \pi [f(x)]^2 dx = \pi \int_1^\infty \frac{1}{x^2}\, dx.$$



The solid of revolution can be generated by rotating an infinite area about the x-axis.

Solution part 2

- Then we have

$$\begin{aligned} V &= \pi \int_1^{\infty} \frac{1}{x^2}\,dx = \pi \lim_{t\to\infty} \int_1^{t} \frac{1}{x^2}\,dx && \text{Rewrite as a limit.} \\ &= \pi \lim_{t\to\infty} \left(-\frac{1}{x}\right)\Bigg|_1^t && \text{Find the antiderivative.} \\ &= \pi \lim_{t\to\infty} \left(-\frac{1}{t}+1\right) && \text{Evaluate the antiderivative.} \\ &= \pi. \end{aligned}$$

- The improper integral converges to π. Therefore, the volume of the solid of revolution is π.

Chapter Opener: Traffic Accidents in a City

Probability Theory:

- If accidents occur at a rate of one every 3 months, then the probability that the time between accidents is between a and b is given by

$$P(a \leq x \leq b) = \int_a^b f(x)\,dx,$$

where

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 3e^{-3x} & \text{if } x \geq 0 \end{cases}$$

Chapter Opener: Traffic Accidents in a City

Solution:

- We need to calculate the probability as an improper integral:

$$\begin{aligned} P(X \geq 8) &= \int_8^\infty 3e^{-3x}\,dx \\ &= \lim_{t\to\infty} \int_8^t 3e^{-3x}\,dx \\ &= \lim_{t\to\infty} \left(-e^{-3x}\right)\Big|_8^t \\ &= \lim_{t\to\infty} \left(-e^{-3t} + e^{-24}\right) \\ &= e^{-24} \approx 3.8 \times 10^{-11}. \end{aligned}$$

- The value 3.8×10^{-11} represents the probability of no accidents in 8

Evaluating an Improper Integral over an Infinite Interval

Evaluate $\displaystyle\int_{-\infty}^{0} \frac{1}{x^2+4}\,dx$**. State whether the improper integral converges or diverges.**

$$\int_{-\infty}^{0} \frac{1}{x^2+4}\,dx = \lim_{t\to-\infty} \int_{t}^{0} \frac{1}{x^2+4}\,dx$$ Rewrite as a limit.

$$= \lim_{t\to-\infty} \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\Bigg|_{t}^{0}$$ Find the antiderivative.

$$= \frac{1}{2}\lim_{t\to-\infty}\left(\tan^{-1}(0) - \tan^{-1}\left(\frac{t}{2}\right)\right)$$ Evaluate the antiderivative.

$$= \frac{\pi}{4}.$$ Evaluate the limit and simplify.

- The improper integral converges to $\frac{\pi}{4}$.

Evaluating an Improper Integral over $(-\infty, \infty)$

Evaluate $\displaystyle\int_{-\infty}^{\infty} xe^x\,dx$. State whether the improper integral is convergent or divergent.

Solution:

- Start by splitting up the integral:

$$\int_{-\infty}^{\infty} xe^x\,dx = \int_{-\infty}^{0} xe^x\,dx + \int_{0}^{\infty} xe^x\,dx.$$

- If either $\int_{-\infty}^{0} xe^x\,dx$ or $\int_{0}^{\infty} xe^x\,dx$ diverges, then $\int_{-\infty}^{\infty} xe^x\,dx$ diverges. Compute each integral separately.

Continued

Solution (continued): For the first integral,

$$\int_{-\infty}^{0} xe^x\,dx = \lim_{t\to-\infty}\int_{t}^{0} xe^x\,dx = \lim_{t\to-\infty}(xe^x - e^x)\Big|_t^0$$
$$= \lim_{t\to-\infty}(-1 - te^t + e^t) = -1.$$

The first improper integral converges. For the second integral,

$$\int_{0}^{\infty} xe^x\,dx = \lim_{t\to\infty}\int_{0}^{t} xe^x\,dx = \lim_{t\to\infty}(xe^x - e^x)\Big|_0^t$$
$$= \lim_{t\to\infty}(te^t - e^t + 1) = \infty.$$

Thus, $\int_0^\infty xe^x\,dx$ diverges. Since this integral diverges, $\int_{-\infty}^{\infty} xe^x\,dx$ diverges as well.

Evaluating an Improper Integral

Evaluate $\int_{-3}^{\infty} e^{-x}\,dx$**. State whether the improper integral converges or diverges.**

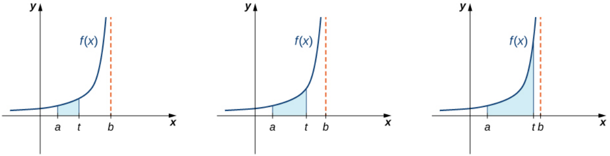
Answer: e^3, converges

Hint: $\int_{-3}^{\infty} e^{-x}\,dx = \lim_{t\to\infty} \int_{-3}^{t} e^{-x}\,dx$

Understanding Integrals with Infinite Discontinuities

In mathematical analysis, the concept of integration is vital for understanding the accumulation of quantities over intervals. However, when dealing with functions that exhibit infinite discontinuities within the interval of integration, a nuanced approach is required.

- Consider an integral of the form: $\int_a^b f(x)\,dx$
- $f(x)$ is continuous over $[a, b)$ but discontinuous at b.
- Let's examine the behavior of this integral as t, the upper limit of integration, approaches b.
- Since $f(x)$ remains continuous over $[a, t]$ for all t satisfying $a < t < b$, the integral $\int_a^t f(x)\,dx$ is well-defined for such values of t.
- We define: $\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx$ provided this limit exists.



Definition

Let $f(x)$ be continuous over $[a, b)$. Then,

$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx.$$

Let $f(x)$ be continuous over $(a, b]$. Then,

$$\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx.$$

If the limit exists, then the improper integral is said to converge. If the limit does not exist, then the improper integral is said to diverge.

Definition

If $f(x)$ is continuous over $[a, b]$ except at a point c in (a, b), then we define $\int_a^b f(x)\,dx$ as

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx,$$

provided both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ converge. If either of these two integrals diverges, then $\int_a^b f(x)\,dx$ diverges.

Integrating a Discontinuous Integrand

Evaluate $\int_0^4 \frac{1}{\sqrt{4-x}}\,dx$, if possible. State whether the integral converges or diverges.

Solution: The function $f(x) = \frac{1}{\sqrt{4-x}}$ is continuous over $[0, 4)$ and discontinuous at 4. Using the above definition, we rewrite $\int_0^4 \frac{1}{\sqrt{4-x}}\,dx$ as a:

$$\int_0^4 \frac{1}{\sqrt{4-x}}\,dx = \lim_{t\to 4^-} \int_0^t \frac{1}{\sqrt{4-x}}\,dx \qquad \text{Rewrite as a limit.}$$

$$= \lim_{t\to 4^-} \left(-2\sqrt{4-x}\right)\Big|_0^t \qquad \text{Find the antiderivative.}$$

$$= \lim_{t\to 4^-} \left(-2\sqrt{4-t}+4\right) \qquad \text{Evaluate the antiderivative.}$$

$$= 4. \qquad \text{Evaluate the limit.}$$

Because the limit exists, the improper integral converges.

Integrating a Discontinuous Integrand

Evaluate $\int_0^2 x\ln(x)\,dx$**. State whether the integral converges or diverges.**

Solution: Since $f(x) = x\ln(x)$ is continuous over $(0, 2]$ and is discontinuous at zero, we can rewrite :

$$\int_0^2 x\ln(x)\,dx = \lim_{t\to 0^+}\int_t^2 x\ln(x)\,dx \quad \text{Rewrite as a limit.}$$

$$= \lim_{t\to 0^+}\left(\frac{1}{2}x^2\ln(x) - \frac{1}{4}x^2\right)\Big|_t^2 \quad \text{Evaluate using integr}$$

$$= \lim_{t\to 0^+}\left(2\ln(2) - 1 - \frac{1}{2}t^2\ln(t) + \frac{1}{4}t^2\right). \quad \text{Evaluate the antideri}$$

$$= 2\ln(2) - 1.$$

The improper integral converges.

Integrating a Discontinuous Integrand

Evaluate $\int_{-1}^{1} \frac{1}{x^3}\,dx$. State whether the improper integral converges or diverges.

Solution: Since $f(x) = \frac{1}{x^3}$ is continuous at every point of $[-1, 1]$ except zero, we use the corresponding definition to write

$$\int_{-1}^{1} \frac{1}{x^3}\,dx = \int_{-1}^{0} \frac{1}{x^3}\,dx + \int_{0}^{1} \frac{1}{x^3}\,dx.$$

Our integral converges if both integrals on the right converge. If either of the two integrals on the right diverges, then the original integral diverges as well. Begin with $\int_{-1}^{0} \frac{1}{x^3}\,dx$:

Solution

$$\int_{-1}^{0} \frac{1}{x^3}\,dx = \lim_{t\to 0^-} \int_{-1}^{t} \frac{1}{x^3}\,dx$$ Rewrite as a limit.

$$= \lim_{t\to 0^-} \left(-\frac{1}{2x^2}\right)\Bigg|_{-1}^{t}$$ Find the antiderivative.

$$= \lim_{t\to 0^-} \left(-\frac{1}{2t^2} + \frac{1}{2}\right)$$ Evaluate the antiderivative.

$$= -\infty.$$ Evaluate the limit.

Therefore, $\int_{-1}^{0} \frac{1}{x^3}\,dx$ diverges, and hence $\int_{-1}^{1} \frac{1}{x^3}\,dx$ diverges regardless of the behavior of $\int_{0}^{1} \frac{1}{x^3}\,dx$.

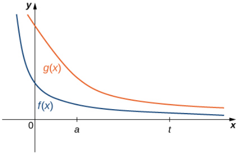
Evaluating an Improper Integral

Evaluate $\int_0^1 \frac{1}{(1-x)^{3/2}}\,dx$**. State whether the integral converges or diverges.**

Answer: ∞, diverges

Comparison Property for Integrals

To see this, consider two continuous functions $f(x)$ and $g(x)$ satisfying $0 \leq f(x) \leq g(x)$ for $x \geq a$. In this case, we may view integrals of these functions over intervals of the form $[a, t]$ as areas. By the comparison property for definite integrals, we have the relationship: $0 \leq \int_a^t f(x)\,dx \leq \int_a^t g(x)\,dx$ for $t \geq a$.



If $0 \leq f(x) \leq g(x)$ for $x \geq a$, then for $t \geq a$,

$$\int_a^\infty f(x)\,dx \leq \int_a^\infty g(x)\,dx.$$

Thus, if $\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx = \infty$, then

$$\int_a^\infty g(x)\,dx = \lim_{t\to\infty} \int_a^t g(x)\,dx \geq \lim_{t\to\infty} \int_a^t f(x)\,dx = \infty$$

as well.

Comparison Theorem

Let $f(x)$ **and** $g(x)$ **be continuous over** $[a, \infty)$. **Assume that** $0 \le f(x) \le g(x)$ **for** $x \ge a$.

- If $\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx = \infty$, then $\int_a^\infty g(x)\,dx = \lim_{t\to\infty} \int_a^t g(x)\,dx = \infty$.
- If $\int_a^\infty g(x)\,dx = \lim_{t\to\infty} \int_a^t g(x)\,dx = L$, where L is a real number, then $\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx = M$ for some real number $M \le L$.

Applying the Comparison Theorem

Use the comparison theorem to show that $\int_1^\infty \frac{1}{xe^x}\,dx$ converges.
Solution:

- The integrand is continuous over $[1, \infty)$ and for $x > 1$:

$$0 \le \frac{1}{xe^x} \le \frac{1}{e^x} = e^{-x}$$

- So if $\int_1^\infty e^{-x}\,dx$ converges, then so does $\int_1^\infty \frac{1}{xe^x}\,dx$.
- To evaluate $\int_1^\infty e^{-x}\,dx$, first rewrite it as a limit:

$$\int_1^\infty e^{-x}\,dx = \lim_{t\to\infty} \int_1^t e^{-x}\,dx = \lim_{t\to\infty} \left(-e^{-x}\right)\Big|_1^t$$
$$= \lim_{t\to\infty} \left(-e^{-t} + e^1\right) = e.$$

- Since the limit is finite, $\int_1^\infty e^{-x}\,dx$ converges, and hence, by the comparison theorem, so does $\int_1^\infty \frac{1}{xe^x}\,dx$.

Applying the Comparison Theorem

Use the comparison theorem to show that $\int_1^\infty \frac{1}{x^p}\,dx$ diverges for all $p < 1$.

Solution:

- First, note that $\frac{1}{x^p}$ is continuous over $[1, \infty)$.
- If $p < 1$, then $\frac{1}{x} \leq \frac{1}{x^p}$ for all $x \in [1, \infty)$.
- We already showed that $\int_1^\infty \frac{1}{x}\,dx = \infty$.
- Therefore, by the comparison theorem, $\int_1^\infty \frac{1}{x^p}\,dx$ diverges for all $p < 1$.

Applying the Comparison Theorem

Use the comparison theorem to show that $\int_e^\infty \frac{\ln(x)}{x}\,dx$ **diverges.**
Hint:

$$\frac{1}{x} \leq \frac{\ln(x)}{x} \quad \text{on} \quad [e, \infty)$$

Applying the Comparison Theorem

Use the comparison theorem to show that $\int_1^\infty \frac{1}{x^p}\,dx$ diverges for all $p < 1$.

Solution:

- First we note that $\frac{1}{x^p}$ is continuous over $[1, \infty)$. If $p < 1$, then $\frac{1}{x} \leq \frac{1}{x^p}$ for all $x \in [1, \infty)$.
- We already showed that $\int_1^\infty \frac{1}{x}\,dx = \infty$. Therefore, by the comparison theorem, $\int_1^\infty \frac{1}{x^p}\,dx$ diverges for all $p < 1$.

Key Concepts and Equations

Key Concepts

- Integrals of functions over infinite intervals are defined in terms of limits.
- Integrals of functions over an interval for which the function has a discontinuity at an endpoint may be defined in terms of limits.
- The convergence or divergence of an improper integral may be determined by comparing it with the value of an improper integral for which the convergence or divergence is known.

Key Equations

$$\int_a^{\infty} f(x)\,dx = \lim_{t\to\infty} \int_a^{t} f(x)\,dx$$

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{t\to-\infty} \int_t^{b} f(x)\,dx$$

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_0^{\infty} f(x)\,dx$$